FINITE DECONVOLUTION DATA RECOVERY SCHEMES FOR BLOCK DATA COMMUNICATION OVER DISPERSIVE CHANNELS

by

POONACHA P G

E E
1986
P
POO
FIN

DEPARTMENT OF ELECTRICAL ENGINEERING

INDIAN INSTITUTE OF TECHNOLOGY KANPUR
OCTOBER 1986

FINITE DECONVOLUTION DATA RECOVERY SCHEMES FOR BLOCK DATA COMMUNICATION OVER DISPERSIVE CHANNELS

A Thesis Submitted
In Partial Fulfilment of the Requirements
for the Degree of

DOCTOR OF PHILOSOPHY

by
POONACHA P G

to the
DEPARTMENT OF ELECTRICAL ENGINEERING

INDIAN INSTITUTE OF TECHNOLOGY KANPUR
OCTOBER 1986

EE-1986-D-POO-FIN

Dedicated to
all those who feel bad about me at
least once in their life time

   We say that the one and many become identified in our
propositions and that now as in time past they run about
together in every sentence which is uttered and that this
union of them will never cease and is not now beginning
but is as I believe an ev rlasting quality of propositions
themselves which never grows old But any young man when
he first tastes these subtleties is delighted and fancies
that he has found a treasure of wisdom in the first
enthusiasm of his joy he leaves no stone or rather no
thought unturned now rolling up the many into the one and
kneading them together now unfolding and dividing them he
puzzles himself first and above all and then he proceeds to
puzzle his neighbors whether they are older or younger or
of his own age — that makes no difference neither fath r
nor mother does he spare no human being who has ears is
safe from him hardly even his dog and a barbarian would
have no chance of escaping him if an interpreter could only
be found

                    Socrates (The dialogues of Plato
                       Jowett vol III Philebus p 566

CERTIFICATE

Certified that this work FINITE DECONVOLUTION DATA
RECOVERY SCHEMES FOR BLOCK DATA CO MUNICATION OVER DISPERSIVE
CHANNELS by Mr P G Poonacha has been carried out under my
supervision and that this has not been submitted elsewhere for
a degree

(M U SIDDIQI)
Department of Electrical Engineering
Indian Institute of Technology
Kanpur

October 1986

ACKNOWLEDGEMENTS

In trying to understand my limitations in order to overcome them if possible I have spent very useful five and a half years at IIT Kanpur in the pleasant company of generous wise and critical teachers friends and wellwishers who have constantly encouraged me to grow and benefit from all possible sources To all of them I express my deep gratitude I feel indebted to this great institution for its free academic environment and to those faculty members who are striving hard to maintain that freedom intact

I feel happy to express my deepest sense of gratitude to my teacher and thesis advisor M U Siddiqi for giving me unconditional freedom to learn anything that I wished I am overwhelmed by his ability to understand and guide a guy of my nature and abnormality This thesis owes a great deal to his patience and critical remarks at all the stages of work He went through the entire manuscript many times in his inimitable way improving the contents and style in many places without losing sight of the overall objective of the thesis I am deeply moved by his genuine concern in my personal growth without expecting anything in return

I wish to express heartfelt thanks to my teachers P R K Rao V P Sinha S J Prabhu and K R Sarma for their affection timely advises many interesting and useful discussions and more than anything for allowing me to speak freely

without getting a feeling of inferiority complex   They have not only helped me a great deal in sharpening my professional skills but also played decisive roles in forming my values and shaping my conduct   My indebtedness to each one of them is more than words of thanks can ever convey

Sundar Rajan Govinda Rajan Uday Madhu Rajhu Shastry Ashok have put themselves to a lot of trouble over helping me with this thesis   I acknowledge my indebledness to them   I don t have words to thank them and a large number of friends who have made my stay t IIT Kanpur all these years a memorable pleasant experience

It gives me great pleasure to thank my close friend C M Abraham for effici nt n at and careful yping and his family for tolerating my visits during odd hours of the day Thank you Abraham

I also wish to thank Qasim Husain for ne tly drawn figures Tiwari and Gangaram for efficient cyclostyling work and Ahmed Binding Centre for neat binding work

Finally whatever little I have done so far owes more than anyone to my d ar parents and sisters   I am eternally indebted to them for never valuating me in terms of marks grades position money and name   Due to thi attitude my home has always been a wonderful place for me to stay and enjoy   Thank you Appa and Amma for always affectionately refusing to lose arguments by saying God giv s you energy to riticise God

Poor Guy

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGUPES

LIST OF SYMBOLS AND ABBREVIATIONS

$\|\|A\|\|_e$	Euclidean norm of a matrix A
BDT	Block data transmission
BTQF	Banded Toeplitz quadratic function
CSDT	Continuous serial data transmission
DF	Decision feedback
DFE	Decision feedback equalizer
DFT	Discrete Fourier transform
DR	Data recovery
d_{min}	Minimum distance
FFT	Fast Fourier transform
$F(\underline{x})$	Function of n variables where $x = (x_1 \; x_2 \quad x_n)^T$
g	Number of ISI components
GE	Gaussian elimination
GF(2)	Galois field of two elements 0 and 1
H	Channel convolution matrix
H_k	k Circulant completion of H
$h_i \; 0 \leq i \leq g$	Channel parameters
H^{LS}	Least squares inverse of H
$h(t)$	Channel impulse response
$H(z)$	z-transform of the autocorrelation function of $h(t)$
H_1	Circulant completion of H
H_{-1}	Skew-Circulant completion of H
H^+	Moore Penrose inverse of H

ISI   Intersymbol interference

LS   Least squares

MD   Maximum distance

MMD   Maximum minimum distance

ML   Maximum likelihood

ML-RDA   Maximum likelihood recursive deconvolution algorithm

MSE   Mean square error

N_o Noise variance

PAM   Pulse amplitude modulation

PBF   Pseudo Boolean function

$(PER)_i$ Probability of ith data symbol being in error

Pe   Probability of error

$Q(x)$ $= \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_x^\infty e^{-y^2/2}\, dy$

$r(A)$ Rank of a matrix A

SDA   Steepest descent algorithm

SNR   Signal-to-noise ratio

$\{S_k\}$ State sequence

T_c Channel data rate in symbols per second

T_h Channel impulse response duration

T_s Source data rate in symbols per second

w_i Sequence of independent Gaussian random variables with mean zero and variance $N_o/2$

VA   Viterbi algorithm

$\{x_i\}$ Sequence of statistically independent data symbols

$X(z)$ z-transform of the autocorrelation function of $h(t)$

$y_k\}$ Output sequence

η_k ν-Circulant matrix with first row (0 0 0   k)

Poonacha P G   Ph D
Department of Electrical Engineering
Indian Institute of Technology Kanpur
October 1986

FINITE DECONVOLUTION DATA RECOVERY SCHEMES FOR
BLOCK DATA COMMUNICATION OVER DISPERSIVE CHANNELS

In this thesis we study several data recovery (DR) schemes
for block data transmission (BDT) over finite impulse response
channels in the presence of noise In block data transmission
due to Clark the input sequence is partitioned into blocks
which are separated by guard zones of appropriate duration such
that the channel output blocks do not overlap   Since the
channel is assumed to be linear its output can be represented
as convolution of the input data values and the channel impulse
response   The operation of recovering data from the nonover-
lapping blocks at the receiver is then essentially that of
finite deconvolution in the presence of noise

We formulate the deconvolution problem as minimization of
a banded Toeplitz quadratic function with data as unknown varia-
bles   When the unknown variables are real minimization of the
quadratic function is straightforward   However when the varia-
bles take values from a finite alphabet as in the case of data
transmission it is difficult to provide computationally attra-
ctive solutions   The solution to the deconvolution problem is

optimal or suboptimal depending on the assumptions made on the
unknown variables and the strategy adopted for solving the pro-
blem

We study maximum likelihood decision feedback and least
squares deconvolution methods in detail Among these schemes
maximum likelihood scheme is an optimal deconvolution scheme
while decision feedback and least square deconvolution schemes
are suboptimal Our main concern throughout is to understand
various theoretical and computational issues involved in
developing DR scheme based on these methods

Salient features of the DR schemes based on the above-
mentioned deconvolution methods are briefly described in the
following paragraphs

1   Maximum Likelihood Deconvolution Scheme

We first derive a maximum likelihood (ML) recursive
deconvolution algorithm (ML-RDA) based on Mitten s lemma which
provides conditions for optimal decomposition of multistage
decision processes This algorithm performs essentially the
job of ML sequence estimation based on Viterbi algorithm
However the derivation of the algorithm does not require the
use of state representation and explicit invocation of Bellman s
principle of optimality which are essential in deriving Viterbi
algorithm The approach taken provides better insight into the
ML deconvolution problem compared to the conventional VA appli-
cation We have shown that if the dat v lues are assumed to

be real variables the recursive algorithm reduces to the familiar Gaussian elimin tion algorithm for solving a set of linear equations We can therefore view ML deconvolution algorithm as a generalized Gaussian elimination algorithm for minimizing banded Toeplitz quadratic objective functions

In the case of binary data we show that the ML RDA reduces to a special case of the algorithm given by Hammer et al for minimizing pseudo Bool an functions We further show that in the binary case the problem of reducing storage and computational complexity of ML RDA involves determination of minimal sum of products form of Boolean functions of several variables Therefore fast algorithm available for such purpose can be used to speed up computations and to reduce storage requirements Finally we show that the recursive algorithm derived for minimizing pseudo Boolean functions can also be used for ML-decoding of binary block codes Usefulness of the method is illustrated with the help of examples However much remains to be done in this direction

2 Decision Feedback Deconvolution Schemes

We next consider the possibility of incorporating decision feedback (DF) in DR schemes for BDT with the objective of achieving good performance without going in for ML-deconvolution which may not be economical in some practical situations

DF-DR schemes consist of a forward filter for minimizing the effect of future symbols a nonlinear decision device and a backward filter for eliminating the effect of past symbols This scheme may be viewed as Gaussian elimination method of solving linear equations using forward elimination and backward substitution with a nonlinear decision device interposed bet ween forward elimination and back ubstitution tages

For deriving DF-DR sch mes we first minimize the banded Toeplitz quadratic function assuming that the data values are real This leads to the solution of banded Toeplitz system of linear equations We then derive three DF-DR chemes for BDT The first scheme is based on Cholesky decomposition of positive definite matrices This scheme however does not take noise v riance into account The second DF-DR scheme takes noise variance into account and is based on minimum mean square error criterion The third DF-DR scheme is based on minimum property of Fourier expansions over finite dimensional Hilbert spaces This scheme is useful when the channel autocorrelation matrix is illconditioned and when no knowledge of the noise variance is available

Finally we derive a DF-DR scheme by modifying Austin s detection theory approach We show that the modified Austin s method can also be viewed as Gaussian elimination method with a nonlinear decision device

Least Squares Deconvolution Schemes-Direct App oach

We then undertake a ritical stuoy o l ast uares (LS) deconvolution using discrete Fourier transform (DFT) techniques The main f ature of our stuoy is in providing a general treat- m nt to the LS-decon/olution probl m using the notion of square completions of the channel onvolution ma rix We show that ooth LS solution and the con/ ntional DFT apo1oach are particular cases of square completions of the chann l convolution matri) Morcover DFT approach to the d convolution probl m is a particular case of k-Circulant completion of the channel convolution matrix Based on k-Circulant completions we develop two data recovery schemes which are easily implementable and have redu ed computational complexity

Although DFT based deconvolution is widely used condi- tions under which such techniques provide optimal solution to deconvolution problem according to some criterion do not seem to have been explicitly m ntioned Ne investigate condiiions on k Circulant comoletions which provide Lo solution to the decon- volution problem Using the framework of quare completion of matrices we obtain conditions on square completion that pro- vide Moore-Penrose inverse of a full rank matrix Such comple tions are called orthogonal completions Then it is srown that orthogonal k-Circulan comol tions are orthogonal k-Circulant matiices Using this result we show that for all practic l

ases of interest LS solution to the deconvolution problem can
not be obtained via direct inversion of k Circul nt completions

4   Least Squares Deconvolution Scheme - Iterative Approach

Finally we con ider an ite ative d con/olution scheme
using Steepest Descent Algorithm (SDA)   One of the purposes of
this study is to examine the obs∘rv tion of several authors
working with fixed st∘p SDA in the DFT domain that the DFT
domain iteration scheme converges faster th∩n the sample
domain scheme   For this purpose we deriv SD4 in the DFT
domain from three dif rent view points   We show that the DFT
domain scheme does not alter the conv rgence rate   it only
helps in reducing computational complexity at each iteration
However   it is shown that k-Circulant comple cions can be used
to obtain good estimates of the convergence parameter in the
case of some examples   These estimated valu∘s can be used
either in the sample domain or in the DFT domin to accelerate
convergence

Since   the iror performance of each data recovery scheme
strongly depends on the particular channel used   we give an
account of good channels for the ourpose of comoaring the per-
formance of various DR sch mes   Two criteria for good channels
are considered   The first criterionprovide channels having
maximum minimum distanc (f 4D) called ꟿ D-channels   These
channels   re used as benchmarks for comparing the performance
of various DR scheme used on actual channels

Determination of MMD-channels however is a difficult problem Therefore we consider the second class of good channels called maximum distance channels (MD-channels) which maximize the total distance A procedure is given for obtaining MMD-channels via MD-channel characterization For the purpose of comparing actual channels with MMD-channels we also develop a recursive algorithm for computing the minimum distance of a given channel It is shown that some of the MD and MD channels are partial response channels

The performance of some of the data recovery schemes discussed earlier has been evaluated using computer simulation Performance curves showing the variation of probability of error with signal-to-noise ratio are given for a wide range of block lengths and typical channel examples

The simulation study brings out the following merits of BDT over continuous serial data transmission (C nf)

1) If decision feedback idea is used to improve the performance of BDT systems error propagation is limited to almost a block length in the worst case whereas in the case of DF-DR schemes for CSDT error propagation can extend over much longer duration Since in the range of medium signal-to-noise ratios the error propagation is a serious problem with DF DR schemes for CSDT systems BDT with DF DR schemes are to be preferred in such applications

2) With an appropriate choice of the block length BDT can be used to provide better trade-off betw en performance processing complexity and data rate

A drawback of BDT is that the effective data rate over the channel is less compared to CSDT for a given source rate   The loss in data rate may be compensated by transmitting data over the channel at a higher rate   This result in increased inter-symbol interference among the block samples   Our simulation studies how ve   indicate that if the block length is properly chosen the degradation in performance of DR schemes for BDT with increased intersymbol interference will not be significant

CHAPTER 1

INTRODUCTION

In this he s we study several data recovery (DR) scheme
for locl data tra m ssion over linear time dispersive channels
in the pr sence of noise   In block data transmission (BDT) the
input data sequence is p rtitioned into blocks which are separa
ted y guard zone o aproo riate duration such that the channel
output blocks do not overlap   ince the han el is assumed to
be linear its outputs can be epresented as convolution of the
input data values and th ch nel impulse response   The op ra
tion of recovering data from the nonoverlap ing blocks at the
channel output is then essen ially that o finite deconvolution
in the presence of noise tha   j   given th channel impulse
response nd the nonoverlapping hannel output blocks which
have been corrupted by noise th probl m s to recover the data
at the receiver nd in some optimun manner

 Ve formul to the deconvolution p o lem as minimization of
a banded Toeplitz quadratic func ion   th data as unknown varia
bl s   When th unknown variables are r al minimization of the
quadratic function is straightforwa d however wh n the varia-
bl s take values fro a finite alphabe   as in the case of data
 r nsmission it is difficult to provid romput ionally attrac-
ctive solutions   The olution to the d con olution problem is

optimal or suboptimal depending on the assumptions made on the unknown variables and the trategy aopted for solving the problem

We study maximum likelihood a cision f e.lback and least squares decon/olution methods n detail   Among these schemes maximum likelihood (iL) scheme is an optimal deconvolution scheme wnile decision feedback (DF) and least squares (LS) deconvolution schemes a   suboptimal   Our ma n concern through-out is to understand va ious theoretical and computational issues involved in developing DR schemes based on these methods

1 1 MOTIVATIO '

Design and evaluation of DR sch nes for long strincs of data transmitted over time dispersiv chann ls in the presence of roise is an importan topic of study in the area of digital communication   Devclopment of v rious data transmission and recovery chene which date back to the pion cring work of Nyqui t [1] and which continue to draw a great deal of atten-tion even today has been primarily iotivated by a de ire to increase the d ta rate   Thi is refl cted in the design of most of the available data transmission and recovery schemes In this connection continuous serial data transr ission (CSDT) happens to be the mo t widely used data transmission scheme in hich a continuous string of serial d a is transmitted over the channel

In the case of data transmission over time dispersive channels due to intersymbol interference (ISI) the signal samples at the receiver input become highly correlated Therefore implementation of optimal DR schemes for CSDT system o on s impractical as they require very large observation intervals which means prohibitively large delays and processing requirements One method of circumventing these problems which has received considerable attention is to make decisions based on truncated observation intervals This strategy reduces delay and processing time at the expense of increased probability of error The trade-off however appears to be one sided from the point of view of the following three major demands a y good data transmission system is expected to satisfy

D1 Minimum probability of error The probability of a detected data symbol being in error must be minimum

D2 Minimum processing time Computational complexity of the data recovery scheme must be minimum and the scheme must be easily implementable

D3 Maximum data rate It must be possible to transmit data values one after another with minimum time gap between adjacent symbols

The foregoing discussion highlights the fact that there is still need for investigating alternate ways of data transmission which will help in striking a better compromise between D1 D2 and D3

An alternate method of reducing delay and proce sing
r quirement without sacrifying error peiformance i to reduce
he observ tion inter/al by aopropriatel) ch ging the data
tran miss on format Thi may be achieved by u ing BDT format
studied bv Clark [2] - [4] Ie tudy various da+a recovery
schemes for BDT with tne following objectives in m rd

1   A better under tand ng of the possible trade-off between
     performance processing comolexity and data transmission
     rate in the context of BDT
2   A better appreciation of the merit and dem iits of the
     existing DR schemes ior CSDT
3   To critically anal) e the advantages of BDT in situations
     where prooability o ciror performance oi the transmission
     system is more import(ant than achieving maximun data rate
     at the cost of erro/ perfoimance as in the c e of CSDT

1 2 REVICW OF LITERArUPE

     To place our study in a proper perspective we hall first
review the exi ting literature on DR schemes for CsDT and BDT
Our int rest is not to give a detailed account of the histori
cal ae/elopment of v rjous DR schemes Inste d we wish to
highlight some of the key developiient with the help of rol
vant literature It may be pointed at the outset that most of
the work is in connection with CSDT ref rences on BDT are
very few 'Ve reviCw the literature on CSDT first and consider
BDT later

An alternate method of reducing delay and processing requirements without sacrifying error performance is to reduce the observation interval by appropriately changing the data transmission format. This may be achieved by using BDT format studied by Clark [2] - [4]. We study various data recovery schemes for BDT with the following objectives in mind

1   A better understanding of the possible trade-offs between performance, processing complexity and data transmission rate in the context of BDT

2   A better appreciation of the merits and demerits of the existing DR schemes for CSDT

3   To critically analyse the advantages of BDT in situations where probability of error performance of the transmission system is more important than achieving maximum data rate at the cost of error performance as in the case of CSDT

1 2   REVIEW OF LITERATURE

To place our study in a proper perspective we shall first review the existing literature on DR schemes for CSDT and BDT. Our interest is not to give a detailed account of the historical development of various DR schemes. Instead we wish to highlight some of the key developments with the help of relevant literature. It may be pointed at the outset that most of the work is in connection with CSDT, references on BDT are very few. We review the literature on CSDT first and consider BDT later

1 2 1   Data Recovery Schen s fo Continuous Serial Data

Transmission

The literature on DR schemes for CSDT in t e presence of
ISI and noise is ri h with a wid variety of novel pproach s
and related theoretical acvelopments   A number of well written
books are available on this topic   i xts by Bennet and
Davey [5] Lucly balz and W ldon [6] and Proakis [7] provide
excellent coverage of most of the important topics in the
area of CSDT longwith long lists of references   the text book
by Clark [4] oroviaes wide coverage to both CSDT and BDT   The
collection of papers eaitcd by Franks [8] on data communication
provides at a placc tw nty four important works on CSDT along-
with informative and well written long ditorial comments
Important dev lop nents that took place in tho do i in of CSDT
during 1968 73 are discussod with extensive bibliography in
a survey article by Lucky [9]   Most o the r oort d data
r covery schemes for CSDT are either some variant of one of th
following thre schemes or their combinations

1   Linear Data Recovery scheme s

Th first and the most simple one is th linear DR scheme
consi ting of a ma ch d filter nat bod to the channel followed
by a trans/c sal filter whose output is ampl d and used in
making decisions   First s st matic study in this connection
app ars to b du to luf ts [10] [11]   Extending th cla sical

work of Nyquist [1] Tufts [10] has derived an optimal linear receiver structure using minimum mear square error (MSE) criterion As an extension of this work Smith [12] has obtained explicit solution to the optimum receiv r when the transmitted waveform is subjected to an average power constraint The criterion of optimization however remains same as before

Deviating considerably from the previous approaches and u ing detection theory point of view Austin [13] has shown that linear receiver can be derived on the assumption that the data values are indep ndent Gaussian random variables The resulting transversal filter in general is of infinite length However for practical purposes only finit length filters are employed and the tap coefficients are optimized by minimizing the MSE between the filter output at the sampling instant and the actual data value Due to their simplicity finite transversal filters have b en us d extensively for the purpose of auto matic channel qualization [14]-[22] as well as for adaptive qualization [23]-[28]

2 Decision Feedback Data R covery Scherm s

The second DR scheme known as decision fe dback equalizer (DFL) [13] [29] consists of a matched filter sampl r forward filter backward f lter and a nonlinear decision device DFE makes use of p t decisions to improv p rfor ance Howev r as the actual decisions are always prone to rror (xcept possibly

at high signal-to-noise ratio (SNR)) DFE suffers from the
problem of error propagation   DFE structure can be derived on
the assumption that the past symbols are error free and that
the future symbols are independent Gaussian random variables
Using this assumption Austin [13] has derived he structure of
the DFE   The resulting forward filter is in general of infi-
nite length   However   a   in the case o   linear DR schemes   it
is common practice to use only finite length and the tap
co fficients are optimiz d using MSE criterion   DFE has since
been studied from   arious viewpoints   [30]-[37]

3   Maximum Likelihood Data Recovery Scheme

The third DR scheme   known as maximum likelihood DR scheme
is due to Forney [38]   He employed Viterbi algorithm (VA) [39]-
[42]   to alleviate the computational complexity associated with
ML sequence estimation of data   Th ML sequence   stimator
minimi es the probability of sequenc   error when all the
s quences are   qually likely   We wish to note here that an
algorithm similar to VA has been dev loped by Kobayashi [43]
for ML decoding of a particular type of PAM signal before the
publication of the Forn y s now classical work [3]

For ML s quence estima ior the VA requires ntire observa
tion values   For long dat s reams such a sch m   therefore
becomes un   tractive bo h from the point of view of computa-
tional corplexity and storag r quirements   To make the VA

useful decisions ar mad after a fixed numb r of observations
becom s av ilaol this numo r is fixed empirically on th basis
of channel memory with the ssumption that d gradation due o
such premature decision is n jlig ble [30] However since the
perform nce degradation strorgl d pends on the channel and
noise such an assumption n cds c reful examination

Aft r Forne/ s work all for s on ML DR schemes seem to
have been dir c ced towards further reducing the complexity of
VA [4]-[52] Specific lly since the computational complexity
of the VA grow xponenti lly with the channel memory efforts
hav been nad to shorten ne channel impulse response duration
by using prefilters

1 2 2 Data Pecovery chem s for Block D a Transmission

Clark [2] [4] has inves ig t d a number of adaptive dete-
ction proces for BDT syst n In his work the BDi has been
referred to as transmission o o thogonal group of signal
elem nts by v rtue of h fact that at the channel output such
groups ao not overlap The rategy used by him is based on the
recognition th t the det ction of l ments in a group can be
carried out by solving a set or l near simultaneous equations
using iterati /e techniques In particular the adaptive dete-
ction processe studied by him ial o u e of the point Gauss-
Seidel iteration n thoa He ha also considered several modifi-
cations of thi basic detection scheme to take care of the

di crete nature of the data From the extensive simulation worl carried out by him and his group [4] it app ars that such iodific tions lead to efficient and cc t effective DR schemes whose performance is comp raole to L approach Moreover such modifications do not require e cessive number of sequential operation Extending thi approa h Clark and Ghani [53] have studied detection processe for multilevel signalling schemes Simulation resul⁺s present d by them show that good toler nce o noise can be achieved oy such detection proc sses which do not require exces ive number of sequential operations

Befoic wincing up this section we wish to take note of th work o Klein anc /olf [54] in which BDT is considered from a different point of vi w They have investigated certain as p cts of h possibility of desisning a receiver rilter such that the overall response of the channel and the receiver fil-ter resembles a cyclic code generator However this line of thought docs not seem to have been pursued further

1 3 KEY R_SULTS AND OB⌐IV⁴TIOⅠ

As mentioned earli the basic aporoach in our study s to view th dh a recovery proble as a finite leconvolution prooblem in the pr sence of noise alient featur s of the study are briefly described in th following paragraphs

1   Maximum Likelihood Deconvolution Scheme

We fir t derive a ML recursive deconvolut on algori-
thm (ML-RDA) based on Mitten s lemma [55] wh ch provides condi-
t ons for optimal decomposition of multistage decision proce ses
This algorithm performs essentially th job of ML equence
e timation bas d on VA   Howe er the deri ation of the
algorithm does not requir the use of state representation and
explicit invocation of Bellman s principle o opti nality [56]
which are essential in deriving VA   Th approach tak n provides
b tter insight into th ML deconvolution problem compared to the
conventional VA application   We have shown that if th data
values a e assumed to be real variables the recursive algorithm
r duces to the familiar Gaussian elimination algorithm for sol
ving a set of linear equations   We can th refore view ML-
deconvolution algorithm as a gen ralized Gaussian elimination
algor thm for minimizing banded Toeplitz quadratic objective
functions

In th case of binary data we show that the ML-RDA reduces
to a special case of the algorithm given by Hammer et al [57]
for minimizing pseudo Boolean fun tion   We further show that
in th binary case the problem of reducing storage and computa-
onal complexity of ML-RDA involves determination of minimal
sum of products form of Boolean functions of several variables
Therefore fast algo ithms v ilable for such purpose can be

used to speed up computations and to reduce storage requirements
Finally we show that the recursive algori hm derived for mini-
mizing pseudo Boolean functions can also b used for ML-decoding
of binary block codes Usefulness of the methoa is illustrated
with the help of examples However much renains to be done in
this direction

2 Decision Feedbac< Deconvolution Schemes

Je ne×t consider the possibili y of incorporating dec sion
feedbacl in DP schemes for BDT wit i the objecuive of achieving
good performance withojt going in for wL-deconvolution which
may not be economical in some practical situation

DF-DR schemes con ist of a foiward filter for minimizing
the effect of future symbols a nonlinear decision device and a
backward filter for cli mnating the effect of past symbols
This scheme may be v ewed as Gaussian elin ination method ot
solving linc ar equations using forward elimination and backward
substitution with a nonlin ar decision de ∕ice ir terposed bet-
ween forward elimination and back substitution stages

For deriving DF-DR chemes we f rst minimizo the banded
Toeplitz quadratic function assuming that the data values are
real This leads to the solution of banded Toeplitz system of
linc r equatjons Ve hen derive three DF-DR schemes for BDT
The first scheme is b ed on Cholesky decompo ition of positive
definite matrices This schem however does not take noise

variance into account   The second DF DF scheme takes noise
variance into account and i   based on minimum mean square error
criterion   The third DF-DR scheme is based on minimum property
of Fourier expansions over finite dimensional Hilbert spaces
[58]   This scheme is useful when the channel autocorrelation
matrix is illconditioned and when no knowl dge of the noise
variance is available

Finally we derive a DF-DP scheme by modifying Austin s
d ection theory approach [15]   We show that the modified
Au tin s method can lso be viewed as Gaussian el mination
method with a nonlinea deci ion device

3   Least Squares Deconvolution Sch mes--Direct Approach

We then undertake a critical stud/ of least squares decon-
volution using discrete Fourier transform (DFT) techniques   The
main feature of   our study is in providing a general treatment
to the LS-deconvolution problem using the notion of square
completions of the channel convolution m trix   We   how that
both LS solution and the conventional DFT approach are particu-
lar cases of squai completions of the ch nnel convolution
matrix   Moreover DFT approach to the deconvolution problem is
a particular cas of k-Circulan completion o the channel con-
volut on matri)   Based on k-Circulant completions we develop
two data recov ry schemes which are easily impl mentable and
have reduced computational complexity

Although DFT based deconvolution is widely used [59]-[63] conditions under which such techniques provide optimal solution to deconvolution problem according to some criterion o not s em to have been expl_citly mentioned We investigate conditions on k-Circulant completions which provide LS solution to he decon/olution problem Using th framework of square completion of matrices we obtain conditions on scuare comple-ions that provide Moore-Penrose invers of a full rank matrix Such completions are call d orthogonal ompl^tions Then it is shown that orthogonal l-Circulart col letions r orthogonal l-Circulant ma rices Usin this sult w show that for all practical cases of int rest L solu on to the deconvolution pioolem can not be obtained via direct inversion of l-Circulant completion

4   Least Squares Deconvolution Scheme-Iterative Approach

Finally we consi ler an iterative deconvolution scheme using steepest desc nt algorithm (SDA) One of the purposes of thi study is to examine the observation of sev ral authors [21] [22] [64] [66] working with fixed step SDA in the DFT domain that the DFT domain itcration scheme converges faster than the sample domain ch me for this purpose we derive SDA in the DFT domain from three different viewpoints We show th t the DFT domain scheme does not alter th convergence rat it only helps in rcduc ng computational comol xity at each

iteration   However it is shown that k-Circulant completions
can be used to obtain good stimates of the convergen e para-
meter in the case of some examples   Th se estimated values can
be us d ither in the sample domain or in the DFT domain to
acc lerate convergence

Since the error p rformance of each I schem trongly
depencs on the particular channel used we give an account of
good channels for the purpos of comparing the performance of
v rious D scheme   Tvo criteria for good channels are con-
idered   Th first cri erion provide channel having maximum
minimum oistance (MMD) c lled MMD channels   Th se channels
are u ed a be chma ks for comparing the performance of
various DR schemes used on ac al chann ls

D termination of MMD-chann ls however is a difficult
problem   Therefor w consid r the second class of good
channels called maximum aistance channels (D cha nel) which
maximiz the total aistance   A procedure is given foi obtain-
ing MD-channels via MD-channel characterization   For the pur-
po of comparing actual channels with MMD-channels we also
a velop a recursiv algo ithn for computing th minimum dis-
t ice of a given chann l   It is shown th t some o the MMD
aid ID channels are partial re ponse channels

The performanc of some of I data recovery schemes dis-
cussed arlie has been evaluated using compu er simulation

Performance curves showing the variation of probability of
rror with s gnal-to-noise ratio are given for a wide range of
block l ngths and typical channel examples

The simulation study brings out the following merits of
BDT over CSDT

1) If decision feedbacl ide is used to improve the per-
formance of BJ1 systems error propagation is limited to almos
1 block length in the worst c se whereas in the case of DF-DI
scne 1 s for C DT error propagation can ex end over much longer
duration since in the ange o mediun signal-to-noise ratios
the error prop g tion is a serious problen with DF-DR schemes
for CSDT systen DDT with JF DR schemes are to be preferred in
such applications

2) With an appropriate choice of the block length BDT can
b used to provide better trade-off b tween performance proce
ssing comolexity and data rate

A drawback of BDT is that the effect ve data rate over the
channel is less compared to CSDT for a given source rate The
loss in data rate may be compensated by transmitting data ov 1
the channel at a higher rate This results in incr a ed IsI
among the block samples Our simula ion studi s how ver indi-
catc hat if h olock length 1 prope ly chosen he degradation
in perfornance o DR schem for BDT with increased ISI will not
be significant

1 4 OPGANIZATION OF THE THESIS

Be ides this chapter the thesis consists of 8 chapters
Chapter 2 gives a detailed account of the assumption made in
the study chann l model considered a discussion of the BDT
format and problem formulation   In Chapter 3 we develop a re
c irsive algorithm for max mum likelihood deconvolution of data
and study the application of this methodolog/ for decoding or
error correcting binary block codes   Decision f edback DR
schemes are studied in Chapter 4   Application of k-Circulant
matrices for designing cost effectiv DR schemes is studied in
Chapter 5   In Chapter 6 we discuss conditions on k-Ci culants
for providing least squares solution to the data recovery pro
blem   Application of steepest descent algorithm for LS-
deconvolution is dis ussed in Chapter 7   In Chapter 8 we
study the performance of some or the DR schemes developed in
the thesis   Chapter 9 gives conclusions and suggestions for
further research

1 5 NOTATIONS AND CONVENTIONS

In this sec ion we wish to mention some of the notations
and conventions which are common to all the chapters of the
thesis   Pest of he notations and conventions are explained in
a list included after the li t of figures   Lower case letters
with a bar below are used to denote sample domain or time do-
main vec ors   For example $\underline{a}$ $\underline{h}$ $\underline{y}$ are sample domain vectors

Upper case letters with a bar below r present DFT or frequency domain vectors of the corresponding samole or time domain v ctors Upper cas lett rs without a bar below except R and C rep esent matrices I_m and O_{nxn} respec ively denote iden-tity matrix of order m and null natrix of order mxn R^m and C^m denote m-dimensional ieal and comple/ vector spaces respectively 'Ve use $R(\triangle)$ to denote the range spa e of A — the space spanned by the columns of A and N(A) to denote the null space of A— the space of ll v ctors $\underline{x}$ such that $\triangle x = \underline{o}$ Lower case letters without under bar are use to denote variable nd constants Transpose of a matri× B is denoted as B^T If B is a complex matrix B denote complex conjugate transpos of B Similarly B^* d notes complex conjugate transpose of vector $\underline{B}$

$\| \, y \, \|$ derotes the Euclidean norm of the vector $\underline{x}$ given by

$$\| \, \underline{x} \, \| = (\underline{x}^* \, x)^{1/2}$$

$\| \, A \, \|$ denotes norm of natrix A When A is symme-tric $\| \, A \, \| = \max\limits_{1 < i < n} \, | \, \lambda_1 \, |$ where $| \, \lambda_1 \, |$ is the absolute value of the i h eigenvalue of n n n natri× A

Fnd o h proof of theorems is indicated by double star (**) Computational complexity of algorithm is denoted by $O(g(n))$ which is to b read as order of $g(n)$

CHAPTER 2

BLOCK DATA TRANSMISSION

In this chapter we et the stage for the development of later chapters   The assumptions made in th stuly are given in Section 2 1   Discrete model of the chann ls used in the stud/ is discu sed in Section 2 2   Block data transmission form t and some of its details are given in Section 2 3 Finally in Section 2 4 ve how that the ueconvolution problem is equivalent to minim za ion of positive d finite Toeplitz quadratic function with data as unknowns

2 1 ASSUMPTIONS

Consider the data transmission system hown in Figure 2 1 In the figure input data stream to the transmitter filter is a sequence of impulses{ $x_i \delta(t-iT_s)$ }where T_s is the intersymbol dur tion and $\delta(t)$ denotes an unit impulse function x_i denotes the value of the ith transmitted symbol   The transmission path is a linear ba eband channel consi ting of a iodulator bandpass channel and a d modulatoi   Noi e is acded at the output of the channel

For the puipose of study we make following assumptions regarding the data transmission system

Input
data
stream

Output
data
stream

Transmitter
Filter

Transmission
channel

noise

$+$

Receiver
Filter

Data
Recovery
scheme

$t = i T_s$
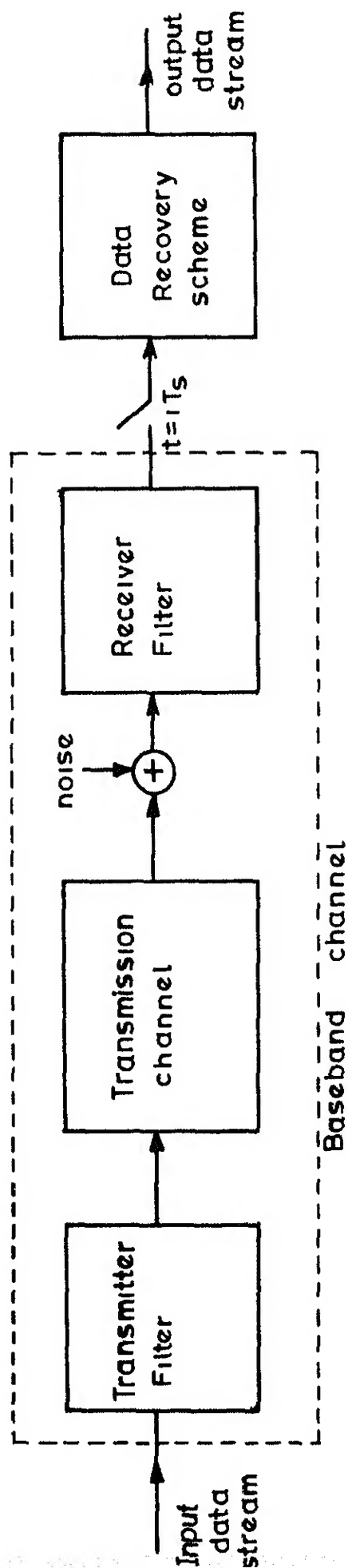
Baseband channel

FIG 21 Block diagram of data transmission system

1 A synchronous data transmission system is assumed That is the receiver has knowledge of the correct pling in tants

2 The ba eband channel is linear and i is characteriz d by ar impulse response h(t) In iost o th prac ical cases channels can be modelled either as time invariant or as lowly ime varying In the case of tin varying channel aata recov ry sch mes must be made adaptive V ho cvei will not s udy this aspect in this the s

3 The channel impuls resounse is of fir e duration This assumption is met b/ mo t o the practic l ch nnels Al hough for a bandlimited channel the impulse response is of irfinite duration in no t of the case ipulse response becomes n gli- c ble after a finite duration

4 Data symbols are statistically independent and take on valu with equal probability from the set $\{\pm 1 \pm 3 \pm 5 \quad \pm(q-1)\}$ where q is an even integer This assumption is rarely met by any practical com unication system Howover it can be r alized t some extent with the help of scramblers at the tran mitter end The effect of scrambling can be removed by descrambling the data symbols at the receiver end (see Clark [4] page 178)

5 Noise samples are statistically independent and each sample is a Gaussian random variabl with zero mean and /ariance $N_0/$ Th white Gaussian noise assumption is widely used as it consiaerably simplifies mathematical analysis and provides worst case performance

It may be noted that although some of these assumptions
may not b valid in prac ic th y provide a us ful f amework
or formulating and analysing the problem

2 2 DISCPETE TIME MODEL FOP ISI CHANNELS

Due to the pr sence of th sampler aft r the receiving
filter (see Figure 2 1) it is conv nient to anal/se the base
band data transmission system with a discrcte time equivalent
nodel   When a finit number of da a ymbols arc transmitted
over finite ISI channels in he presence o whit G ussian
noise it can be shown ⌐33] [7] that he channel od 1 shown
in Figure 2 2 provides an quivalent model for the estimation
of th input sequence without any loss o rror performance
For this to happen transfer function of the transversal filter
I (z) must be chosen such that

$$X(z) = H(z) \ H^{*}(z) \qquad\qquad (2\ 1)$$

whore X(z) is the z-transform of the autocorrclation function
of the channel impulse response h(t) and H(z) is a polynomial
having roots of X(z) which are in ide the unit circle

Clark [3] has consider a chairel responses deriv d in a
d ffcrent manner   H as une that effcctive multipath distor
tion intro luced by the transmission path can be modelled as an
quival nt sampled impulse respcnse of the linear tiansmission
oath   The ov rull snipled r spons of th channel is obtained
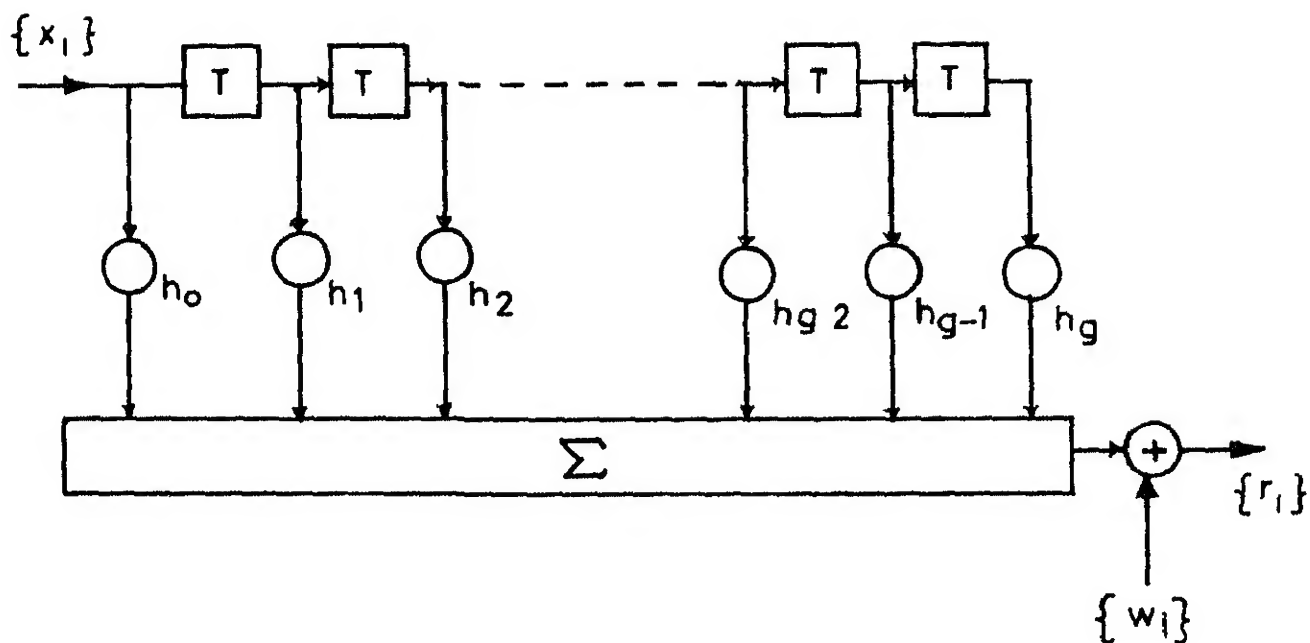
FIG 2 2 Discrete model of finite impulse response
channel

by considering identical transfer functions for bo h transmitter
and receiver filters

For the purpose of study we use the discrete time model for
finite ISI channels sho m in Figure 2   The model consists of
a finit length transver l lt r who e tap coefficients
reflect the ffec o transmitter filter cha nel and receiving
filter Flexibility in the choi e of the tap coefficients to
take care of a w d class of channels including channel varia-
tions (by making the tap coefficients vary with time) nd the
imple manner in which it captures the es en ial idea of ISI
and noise are the major onsiderations for u ing t e model in
the s udy

In Figure 2 2 $\{x_1\}$ denotes a s quence of statistically
independent data s/m ols $\{w_1\}$ denotes a sequence of independent
Gau sian random variables wi th mean zero and variance $N_0/2$
h_1 $1 = 0\ 1$ g are the ch nnel param t s j indicates
the number of ISI components and it depends upon h   data
rate and duration o the channel irpul e response As the data
rate increases y also increases

From the chann l mod l f Figur 2 2 it is easy to see
that ch output sequenc $\{r_1\}$ has the following r presentation

$$r_1 = \sum_{k=0}^{g} h_k\, x_{1-k}\quad w_1 \qquad 1 = 0\ 1\ 2 \qquad (2\ 2)$$

Data recov r/ problem in h case of CSDT systems is
essentially that of solving the   et of equation giv n by
(2 2) for x_1 s In the ca e of long strinjs of data this
becomes a di ficult problcm due to the larg numoer of equa-
tions which are to be solved n h pr sence of noise Block
data transmission rormat which overcomes this proble1 is des-
cribed in the following ection

2 3 BLOCK DAT/ TR1NG ISSION FORMAT

In BDT blocls of data of l ngth m wi h each block
followed by g z ros is transmitved through the ch nnel   If
the source data rate s $1/f_s$ symbols p r socord due to the g
zcro following cach block th ffctive rate of transmis ion
, 11 be $m/(nT_s)$ sy1bols per second where n = m+g On the
other hand no reduction in data rate is xperienced if he data
symbols are transmitted through th channel at a rate of $1/T_c$
symbols per second where T_c is obtained a $\hat{}$ _ n L t g
b chosen such that the channel dispe sion ruration (T_h) coes
not e cccd $(g +1)_{i_c}$ seconds   fhat is

$$(g +1)T_c > T_h \qquad\qquad (2\ 3)$$

Then T_c 1 gi\ r by

$$T_c = n 1_s/(m+g) \qquad\qquad (2\ 4)$$

uos ituting the v lue of T_c from (2 4) n (2 3) we obtain

$$g > m (T_h - T_s)/(mT_s - T_h) , \quad mT_s > T_h \qquad (2\ 5)$$

Due to the presence of g z ros following each transmitted block the channel is forced into zero state before the arrival of a fresh block at the input hus the effect of the channel is confined to blocks of lenj h n where n = m+g Therefore ISI does not propaga e from one block to the ubsequent ones Due to this f ature blocls can be processed ind pend ntly at the receiver to recover data value Note that two buffers will be needed of which one will be receiving the next block hile the other holds th current block for proce sing

Figure 2 3 helps to nake some of the ic s expressed in the previou paragraphs more clear In Figu 2 3a a rather simplistic channel is shown for the purpose of illustration Th anta duration is assumed to be one third of tne duration of the impulse respons Outout of the channel shown in Figure 2 3a to an arbitrarily chosen string of binary (± 1) data is shown in Figure 2 3b Figure 2 3c shows the corresponding overlapping pulse responses Figures 2 3d and 2 3e show the corresponding situation in the case of BDT with increased da a rat over the channel In this case m = 6 q = and n = 10 Moreover w obtain $T_c = 0 6 T_s$ as the data transmission rate over the channel From the Figures 2 d and 2 3e it c n be seen tha the blocls get separated at th channel output

Cer ain features of the inequality (2 5) can be made clear with a graphical repr entat on For this purpose

( ) Ch l p

h(t)

t → T_h
T_h 3T

(b) CSDT R lt t h l tp t th b af

t

( ) CSDT O l pp g f p l t th tp t f th ha l

(d) BDT R lt t hd l tp t th ab
(m 6 g 4 t-06q)

t →

( ) BDT O l pp ng f p l s at th tp t f th ha l

T k
Bl k
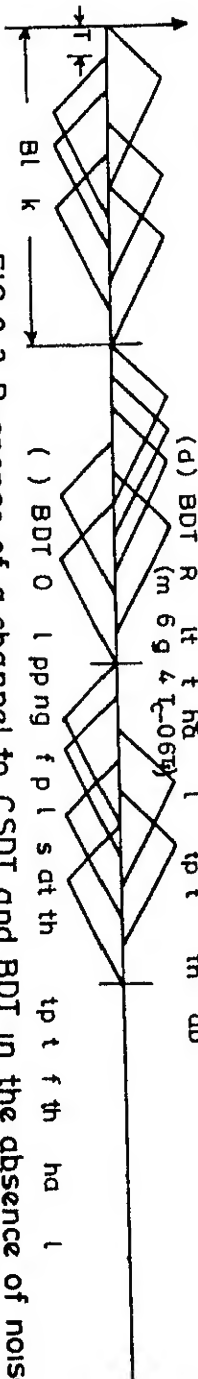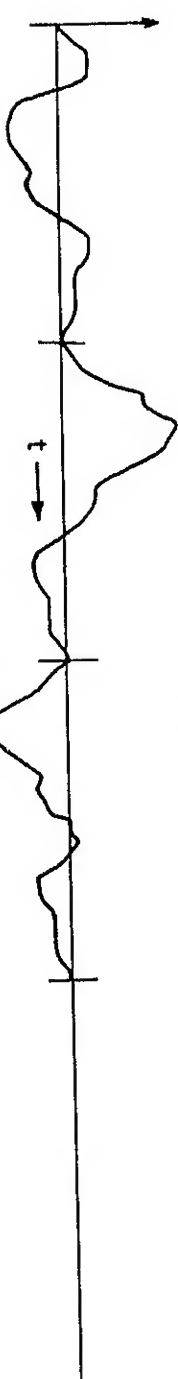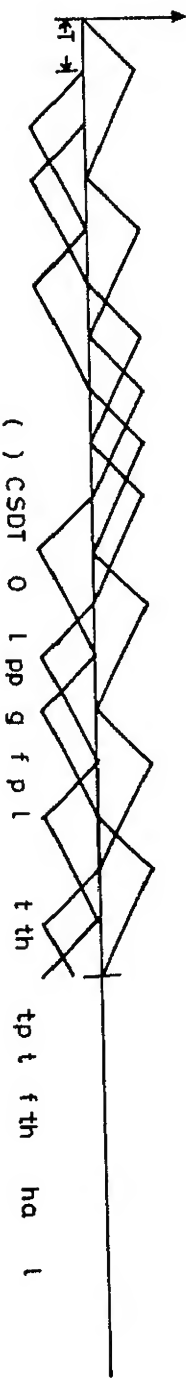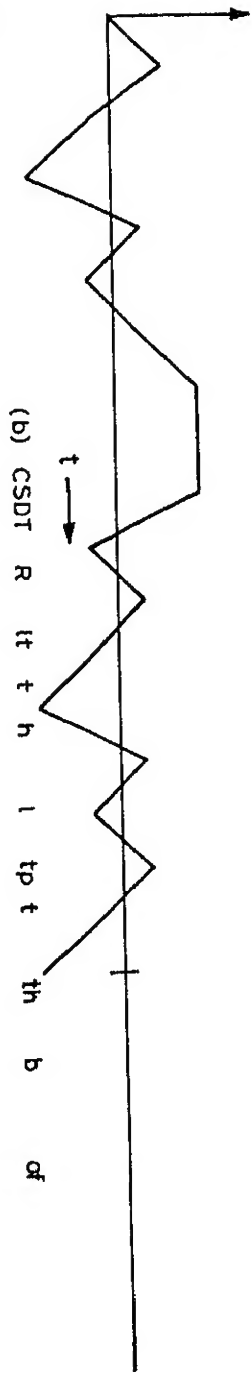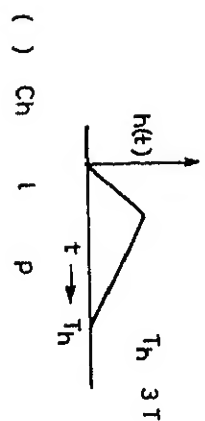
FIG 2 3 Response of a channel to CSDT and BDT in the absence of noise

let $p = T_h/T_s$ Then (2 5) becomes

$$g \; > \; (p-1) \; m/(m-p) \quad m > p \qquad (2\;6)$$

In Figure 2 4 g is plo ted against m ror variovs valuvs of p
wh n g is given by

$$g \; = \; (p\;1) \; m/(m-p) \quad m > p$$

N te that althouqh n an g arc integ rs in Figurc 2 4
in order to avoid oo many ymbols g is plotted as a continuou
function of m for m > p   From the figur it s clear t1at for a
gi ven p m must bc sufficiently large for g   to bc small Under
this condition T_c will b close to T_s Je al o observe that
the requirc value of g lev ls off fast after dropping rapidly
 n th beginning   This m ans that the effcct of ISI among the
memb rs of a block will not become too large if moderate value
of n is choscn

2 4 THE DECO JVOLUTION PROLLLM

In this scction we formulate the d convolution problem
L t h bc an n-l ngth vector with first g elements as the
channel par met rs and res of tn elcmer ts zero   Lct x denote
an m-lcngtl uata vector and r a n length obs rvation vector In
terms of th sc vectors h convolution relationship given by
(2 2) can bc ritten as

$$\underline{r} \approx H \; x + \upsilon \qquad (2\;7)$$
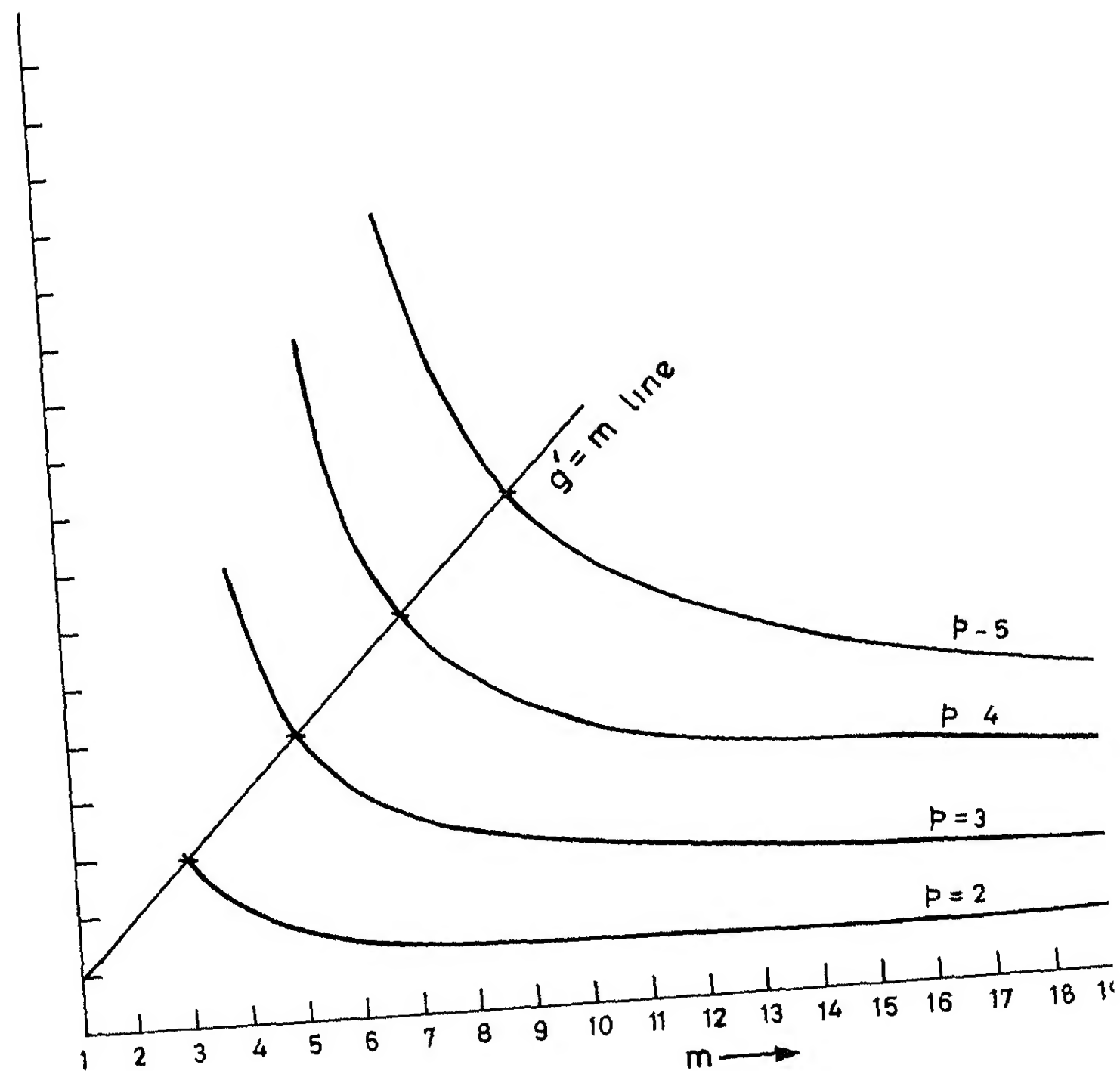
FIG 2 4 Plot of g′ versus m

wh H 1 th convolution matr1   w1th h as t f1r t column

For th sarc o 1llustrat1on H matr1 1s hown b low

$$H = \begin{bmatrix} h_o & 0 & 0 & & 0 \\ h_1 & h_o & 0 & & \\ & h_1 & h_o & & \\ & & h_1 & & 0 \\ h_g & & & & h_o \\ 0 & h_g & & & \\ & 0 & h_g & & \\ 0 & 0 & 0 & & n_g \end{bmatrix}$$

h $(1\ j)$th el n nt of H 1s g1ven by

$$h_{1j} = \begin{cases} h_{|1-j|} & 1 \le j \\ 0 & \text{for } 1 > j \text{ an} |1-j| > g \end{cases}$$

$$1 = 1\ 2\ 3 \qquad n \qquad j = 1\ 2\ 3 \qquad m$$

The problem of d t 1m1n1ng x g1ven r and $\underline{h}$ or f1nd1ng $\underline{h}$

g1v n $\underline{r}$ and $\underline{x}$ 1s call d f1n1te d convolut1on probl m 1n the

pr s nc of no1se

Io problem has received as much attention as the finite
deconvolution problem due o its ppearance in a wide variety
of applications [59]-[61] [67]-[71] However most o the
methods available for dealing with the problem a ume that the
unknown variables are real Whereas in the context of BDT
th dat vector x take di cr te values from a inite alphabet
Therefore in this case / have to deal w th a combinatorial
ae onvolution problem in which the pace oi all possible solu-
tiors do not po sess the nice prop rties of R^n to which real
vectors belong One straightforward method of deconvolution
is t search through all possible solution Ho ver even in
the binary case searching through all possible 2^r vectors
becomes out of hand for moderate values of m Moreover due to
the pr s nc of noise no xact solution is possible in general
The usual app oach tak n under such situations is to choose a
vector x which optimize some suitable performance criterion
Depending on the constraints put on the solution space one
obtains different s lutions to the deconvolution problem in
the presence of noise

In d ta co munications the mos sought after criterion is
the minimization of tl rroi probability For he case of
qually lik ly data vector it can be shown that the maximum
likelihood (ML) seque ce estimation provides mini ium probability
of rror [7] The ML esti ate of x i tht vector which maxi-
mi es the conditional density o(i/x)

Under the assumptions made in section 2.1 the conditional density function of r given a particular data vector x can be shown to be

$$p(r/x) = \prod_{i=1}^{n} \frac{1}{\sqrt{\pi N_o}} \exp\left(\frac{1}{N_o} (r_i - \sum_{l=o}^{g} h_k x_{i-k})^2 \right)$$

it is easy to see that the above density function is maximized by minimizing the quadratic function

$$F(x) = (\underline{r}-H\underline{x})^T (r-H\underline{x})$$

$$= \sum_{i-1}^{m} \sum_{j=1}^{n} x_i x_j a_{|i-j|} - \sum_{i=1}^{m} b_i x_i + \sum_{i-1}^{m} r_i^2 \qquad (2.8)$$

where $a_{|i-j|}$ is the (i,j)th element of $h^T H$ and b_i is the ith element of the vector $2H^T \underline{r}$.

Therefore the deconvolution problem in our case is equivalent to minimization of the quadratic function given by (2.8) subject to the condition that the unknown variables take values from a finite alphabet. Since H is a convolution matrix it is easy to see that $H^T H$ is a symmetric Toeplitz matrix. We wish to note that a square matrix D is called Toeplitz if $d_{ij} = d_{|i-j|}$ where d_{ij} is its (i,j)th element. In addition as g is less than n $H^T H$ is also a band matrix. Thus the deconvolution problem is essentially that of minimizing a banded Toeplitz quadratic function subject to different constraints on the unknown variables depending on various performance criteria

CHAPTER 3

MAXIMUM LIKELIHOOD D CONVOLUTION

In this chapter we use Mitten s lemma [55] to derive a
maximum likelihood recursive deconvolution algorithm (ML-RDA)
which essentially performs the job of Viterbi algorithm (VA)
[38] Mi ten s lemma may be viewed as a more formal statement
of Bellman s principl of optimality [56] and it clearly stipu-
lates the conditions for optimal decomposition of multistage
op imization problems For the sake of easy reference Mitten s
lemma is given in Appendix A

The chapter is organized in th ollowing manner In
Section 3 1 we use Mit en lemma to develop a general recur-
sive algorithm for minimizing decomposable nonlinear function
of several variables A specific case when tre nonlinear fun
ction is a quadr tic function s taken up in Section 3 1 1 It
i shown that minimization of a quadratic function of real
variables using the r cursive algorithm is equivalent to
Gaussian elimination algorithm for solving a set of linear
simultaneous quations The discret version of the algorithm
i discussed in section 3 1 2 The ML-RDA for a banded
Toeplitz quadratic unction (BTQF) is given in Section 3 2
For the purpose of comp 1 on in Sec on 3 3 we discuss VA

for minimizing BTQF Strategies for alleviating storage and computational requirements are discussed in Section 3 4   Application of th se stratcgies to Binary case is considered in detail in Section 3 4 1   Application of the recursive algorithm for decoding of block error correcting code is considered in Section 3 5

3 1 RECURSIVE ALGORITHM FO, IINI IIZATION OF NONLINEAR FUNCTION OF SEVERAL V PIABLES

In order to illustrate th principl of ML RDA we first give a method for minimizing a nonlinear function of n variables Let $F(x_1 \ x_2 \quad x_n)$ bc a function to be minimized We assum that th function is decomposable in the manner indicated below

$$F_1(\underline{x}) = F(x_1 \ x_2 \qquad x_n) = U_1(x_1 \ x_2 \qquad x_n) +$$
$$\qquad \qquad /_1(x_2 \ x_3 \qquad x_n) \qquad \qquad (3 \ 1)$$

Note that V_1 is func ion of $x_2 \ x_3 \qquad x_n$ alone Now suppose that inst ad of minimizing $F(\underline{x})$ with rcspect to $\underline{x}$ ve can minimize U_1 with respect to x_1 first and then minimize the combined function $\min_{x_1} U_1(x_1 \ x_2 \qquad x_n) + V_1(x_2 \ x_3 \qquad x_n)$ with r spect to $x_2 \ x_3 \qquad x_n$ Put differently we wish to compute P_2 where

$$P_2 = \min_{x_2 \ x_n} [V_1(x_2 \ x_3 \qquad x_n) + \min_{x_1} U_1(x_1 \ x_2 \qquad x_n)]$$
$$\qquad \qquad (3 \ 2)$$

At this stage a valid question to ask is whether or not

$$P_2 = \min_{\substack{x_1 \\ x_n}} F_1(x_1\ x_2 \qquad x_n)?$$ If the answer is y s then

eff ctively we have decomposed the n variable minimization

problem into a single variable minimization pioblem followed by

an n-1 variable minimization problem Thus it is essential to

know whether we can distribute min operator as indicated in

(3 2) without affecting the op inum solution

In Lemma 3 1 we show that whenever $F_1(x_1 \qquad x_n)$ is

decomposable in the manner given by (3 1)

$$P_2 = \min_{\substack{x_1 \\ x_n}} F_1(x_1\ x_2 \qquad x_n)$$

which efrectively pioves that the min operator can be distribu-

ted in the manner shown in (3 2) We wish to note that

Lemma ᵤ 1 is a sp cial case of the more general decomposition

result giv n by Mitt n [55] for multivariable optimiz tion

problems A proof o Mitt ns result for two variables may be

found in [56]

Lcmma 3 1   Given (3 1) and (2) $P_2 = \min_{\substack{x_1 \\ x_n}} F_1(x_1\ x_2 \qquad x_n)$

Pioof   Lct $P_1 = \min_{\substack{x_1 \\ x_n}} F_1(x_1\ x_2 \qquad x_n)$

From the definition of minimum value we have

$$P_2 \geq P_1 \qquad\qquad\qquad\qquad (3\ 3)$$

Whenever $U_1(x_1 \, x_2 \qquad x_n) < U_1(x_1 \, x_2 \qquad x_n)$ with

$x_2 \, x_3 \qquad x_n$ fixed it follows from (3 1) (since

$V_1(x_2 \, x_3 \qquad x_n)$ is fixed) that

$$F_1(x_1 \, x_2 \qquad x_n) < F_1(x_1 \, x_2 \qquad x_n) \qquad\qquad (3 \ 4)$$

But for each value of $x_2 \, x_3 \qquad x_n$

$$U_1(x_1 \, x_2 \qquad x_n) = \min_{x_1} U_1(x_1 \, x_2 \qquad x_n) \leq U_1(x_1 \, x_2 \qquad x_n)$$

whe x_1 is the value of x_1 at which $U_1(x_1 \, x_2 \qquad x_n)$

attains minimum value when $x_2 \, x_3 \qquad x_n$ are fixed

Therfore from (3 4) we t

$$F_1(x_1 \, x_2 \qquad x_n) \leq F_1(x_1 \, x_2 \qquad x_n)$$

for all x_1 with $x_2 \, x_3 \qquad x_n$ fixed (since $V_1(x_2 \, x_3 \qquad x_n)$

is fixed)

Hence

$$F_1(x_1 \, x_2 \, x_3 \qquad x_n) \leq \min_{x_1} F_1(x_1 \, x_2 \qquad x_n)$$

Hnc $P_2 = \min_{x_2 \quad x_n} [V_1(x_2 \, x_3 \qquad x_n) + \min_{x_1} U_1(x_1 \, x_2 \qquad x_n)]$

$$= \min_{x_2 \quad x_n} F_1(x_1 \, x_2 \qquad x_n) \leq$$

$$\min_{x_2 \quad x_n} \min_{x_1} F_1(x_1 \, x_2 \qquad x_n) = P_1$$

therefor $P_2 \leq P_1$

From (3) we thus get $P_2 = P_1$ **

Let x_1 be the value of x_1 which minimizes $U_1(x_1 \ x_2 \quad x_n)$ for a fixed $x_2 \quad x_n$ No that x_1 becomes a function of $x_2 \ x_3 \quad x_n$ Let

$$x_1 = g_1(x_2 \ x_3 \qquad x_n)$$

Substituting for x_1 in $F_1(_1 \ x_2 \qquad x_n)$ the minimization problem now reduces to a $n-1$ variable minimization problem

$$\min_{x_2 \quad x_n} \quad F_2(x_2 \ x_3 \qquad x_n) \quad \text{where} \quad F_2(x_2 \ x_3 \qquad x_n) =$$

$$F_1(x_1^{'} \ x_2 \qquad x_n)$$

Assuming again that F_2 is decomposable we can write F_2 as

$$F_2(x_2 \ x_3 \qquad x_n) = U_2(x_2 \ x_3 \qquad x_n) + V_2(x_3 \ x_4 \qquad x_n)$$

Using Lemma 3 1 we can write

$$\min_{x_2 \quad x_n} \quad F_2(x_2 \ x_3 \qquad x_n) = \min_{x_3 \quad x_n} \quad [V_2(x_3 \ x_4 \qquad x_n) +$$

$$\min_{x_2} \quad U_2(x_2 \ x_3 \qquad x_n)]$$

Let x_2 be the value of x_2 which minimizes $U_2(x_2 \ x_3 \qquad x_n)$ Then

$$x_2^{*} = g (x_3 \ x_4 \qquad x_n)$$

The recursive procedure now can be easily written for the ith stage as

$$F_i(x_i\ x_{i+1} \qquad x_n) = F_{i-1}(x_1\ x_2 \qquad x^*_{i-1}\ x_i \qquad x_n)$$

$$\min_{x_i \quad x_n} F_i = \min_{x_{i+1} \quad x_n} [V_i(x_{i+1} \qquad x_n) + \min_{x_i} U_i(x_i \qquad x_n)]$$

$$x_i - g_i(x_{i+1} \qquad x_r)$$

$$i = 1\ 2 \qquad n$$

At the nth stage we obtain

$$\min_{x_n} F_n(x_n) = \min_{x_n} U_n(x_n)$$

Th refore x^*_n is a constant i e g_n is a constant function substituting x^t_n in $g_{n-1}(x_{n-1})$ x_{n-1} can be obtained Continuing this procedure of back substitucion all the x_i $i = 1\ 2$ n are obtained

It is not dif icult to see that the foregoing approach is v ry similar in spirit to the Gaussian elimination method of solving linear inultaneous equations which involves forward limination and back substitution The collection of functions

$$x^*_1 = g_1(x_2\ x_3 \qquad x_4)$$
$$x^*_2 = g_2(x_3\ x_4 \qquad x_n)$$

$$x^*_i = g_i(x_{i+1}\ x_{i+2} \qquad x_n)$$

$$x_n = g_n\ \text{constant}$$

is reminiscent of th upper triangular m tri which results
after forward elimination In view of the foregoing observa-
tion we wish to call the recursiv algorithm dev loped above a
g neralized Gaussiar elimination procedure

In the preceeding algo ithm we implicitly assumed that the
indic ted single variable minimization is lways possible That
is $\min_{y} U_1(x_1 \quad x_n)$ c ist for all i and f xed $x_{i+1} \quad x_n$
$i = 1^1 2 \quad n$ Ho e er this may not alway h possible for
an arbitrarily specifi d nonlinea fur tion In ad dition sin-
gle variable minimi ation itself may be difficult to carry out
N v rtheless this approacn has several advantages in the case
of the ML-deconvolution roblem ormulated in Sec 2 4 Before
discussing that case we wish to cons d r the probl m of mini-
mizing a quadratic func ion of n rea variables as an illustra-
ion of the oregoing recursive algorithm

3 1 1 Real Variable Case

For the sake of simplicity we consider n = 3 Generaliza-
tion to n variables is straightforward Let

$$F_1(x_1 \ x_2 \ x_3) = -2 \sum_{i=1}^{3} x_i b_i + \sum_{i=1}^{3} \sum_{j=1}^{3} x_i x_j a_{ij} + \sum_{i=1}^{3} r_i^2$$

$$\text{and} \quad a_{ij} = a_{ji} \quad \text{for all i j}$$

where b_i a_i and r_i i = 1 2 3 are constrants Carrying out
the decomposition of $_1$ with respect to x_1 w obtain

$$F_1 = U_1(x_1 \ x_2 \ x_3) + V_1(x_2 \ x_3)$$

where $U_1(x_1 \ x_2 \ x_3) = -2x_1 b_1 + 2x_1 \sum_{j=2}^{3} x_j a_{1j} + x_1^2 a_{11}$

x_1^* which minimizes U_1 with $x_2 \ x_3$ fixed is given by

$$x_1 = (b_1 - \sum_{j=2}^{3} x_j a_{1j})/a_{11}$$

we assume that $a_{11} \neq 0$ Substituting x_1^* in F_1 we obtain

$$F_2(x_2 \ x_3) = -2 \sum_{i=2}^{3} x_i b_i + \sum_{i=2}^{3} \sum_{j=2}^{3} x_i x_j a_{ij} + \sum_{i=2}^{3} r_i^2 + (r_1)^2$$

where $b_i = b_i - b_1 \ a_{1i}/a_{11}$ $i = 2 \ 3$

$$(r_1)^2 = r_1^2 - b_1^2/a_{11}$$

and $a_{ij} = a_{ij} \ a_{1i} \ a_{1j}/a_{11}$ $i \ j = 2 \ 3$

low $F_2(x_2 \ x_3) = U_2(x_2 \ x_3) + V_2(x_3)$

where $U_2(x_2 \ x_3) = -2x_2 b_2 + 2x_2 x_3 a_{23} + x_2^2 a_{22}$

therefore $x_2^* = (b_2 - x_3 a_{23})/a_{22}$ $a_{22} \neq 0$

using x_2^t we obtain $F_3(x_3) = -2x_3 b_3 + x_3^2 a_{33} + (r_1)^2 + (r_2)^2 + r_3^2$

minimization of F_3 gives

$$x_3 = b_3/a_{33}$$ $a_{33} \neq 0$

Back substitution of x_3^* gives x_2^* and chain on gives x_1^*

The foregoing example clearly shows that the recur ive algorithm for minimization of a quadratic function or n variables is equivalent to Gaussian elimination technique for solving a set of linear equations

3 1 2 Discrete Variable Ca e

Having di cu ed certain aspects of the multivariable minimization with real variables we now conside the case when he var ables take values from a inite et Without any loss of generality we consider tne quadratic func ion used in the p evious section with the d ferenc that th varia les take values from the set $S = \{-3\ -2\ -1\ 0\ 1\ 2\ 3\}$ Extension to n-variable case and any given finite alph ω t is straightforward

Following the steps as in the real variable cas in the first stage we have to find $\min_{x_1} U_1(x_1\ x_2\ x_3)$ In this case differentiation i not possible Hence we must tabulate values of x_1 which minimize U_1 for fixed x_2 and x_3 The tabulation format is shown in Table 3 1 No of en ries in th first column of th table will be 7^2 corresponding to 7^2 possible values of x_2 and x_3 Moreover for each set of x_2 and x_3 we ut ch ck 7 v lu s of x_1 o decide x_1^t Thus th first stage itself involv s 7^3 evalu tions of U_1 Furth r in general it is not possible to expres $_1^t$ in a closed form as in the case of i l variables Therefore x_1 can not b uostituted in F_1 to ob ain F_2 This implic that we must use h t ble for x_1^t in th second stage In th secona stage we write F_2 as

$$F_2(x_2\ x_3) = J_1^t(x_1\ x_2\ x_3) + U_2(x_2\ x_3) + V_2(x_3)$$

Tabl 3 1 Stage 1

x_2	x_3	x_1^*	U_1
3	-3	-	-
3	-2	-	-
3	-1	-	
	3	-	

Tabl 3 2 Stage 2

x_3	x_2	J_2
-3	-	-
-2	-	-
-1	-	-
0		
1	-	-
2	-	-
3	-	

For each value of x_2 and x_3 w can use U_1 from the table for x_1^t Thu in tne se ond stage we minimze U_1+U_2 $(x_2$ $x_3)$ with r spcct to x_2 by fixing x_3 Th rc aie 7 different values for x_3 rd for each x_3 x_2^t is to be decided on the basis of 7 possible valu s of x_2 Thu the second table will have 7 ntries as shown in Tabl 3 2 In a si ilar fas ion we can carry out the computation required ir stage 3 with the help of Table 3 2 of stage 2

Let us nov xamine the comoutational conplexity for n- variable asc w th ach variable taking q discrete values In the first tage q^n ev luations of U_1 is rcquired Likewise in the ith stage q^{n-i+1} evaluation of $U_1+U_{1-1}^*$ is rcquired On the oth r hand a dir ct scarch method would require q^n evaluation of

th funct on F It thus becomes clear that for the sch^me des-
cribed to become computa ionally efficient the unctional forms
of U_1 U_2 U_n must be much impler than the functional form
of F In otherword U_1 U_2 U_n must have the following
desirable properties

1   U_1 s must be functions of a iall number or variables
compared to the total number of variables involv d in the
minimization problem

2 U_1 s must be easily computabl   By this we near the number
of addition and multiplications needea for evaluating the
function must be s iall

When both the propertie are sat sfied by an objective fun-
c ion it i possible to reduce the number of computations and
storage requirements substantially using the foregoing recur-
sive algorithm compared to the direct enumeration approach It
may how ver be noted that storage requirement is still large
compared to the enumeration approach

The obj ctive function in the cas of ML-deconvolution
probl m formulat d in Section 2 4 is a special type of quadra-
tic function called banded Toeplitz quadratic function Such
a quad itic function me ts th above mentioned requirements In
the next section we derive a hL-RDA or this case

3 2 RECURSIVE ALGORITHM FOR BANDED TOEPLITZ QUADRATIC FUNCTIONS

We recall from Section 2 4 that the function

$$F(x_1 \; x_2 \quad x_m) = 2 \sum_{i=1}^{m} x_i b_i + \sum_{i=1}^{m} \sum_{j=1}^{m} x_i x_j a_{|i-j|} + c$$

where $a_{|i-j|} = 0$ for $|i-j| > g \quad g < m$

is called banded Toeplitz quadratic function (BTQF) for the
matrix formed by $a_{|i-j|}$ as elements is a banded Toeplitz
matrix

Let $F_1(\underline{x}) = F(x)$ for the sake of uniformity F_1 can be
decomposed as

$$F_1 = -2x_1 b_1 + 2y_1 \sum_{i=2}^{g 1} x_i a_{|i-1|} + x_1^2 a_0$$

$$+ [-2 \sum_{i=2}^{m} i b_i + \sum_{i=2}^{m} \sum_{j=2}^{m} x_i x_j a_{|i-j|}]$$

$$= U_1(x_1 \; x_2 \quad x_{g+1}) + V_1(x_2 \; x_3 \quad x_m)$$

Note that U_1 is only a function of g+1 variables Further
U_1 is a function of consecutive g+1 variables rather than g+1
arbitrary variables as it may happen in the case of a general
objective function This is a special property of BTQF In
addition the form of U_1 at every stage is same Thus the same
type of computation procedure is used in every stage Through
the following example we have tried to illustrate various
aspects of the algorithm The example is from [11]

xample 3 1

Let $F_1(x_1 \ x_2 \qquad x_7) = -2 \sum_{i=1}^{7} Y_i b_i + \sum_{i-1} \sum_{j=1}^{7} x_i x_j a_{|i-j|}$

where $b_1 = 1\ 0$ $b_5 = 1\ 5$ $a_0 = 1$ $a_1 = 0\ 0$ $a_2 = -0\ 25$

$b_2 = 2\ 0$ $b_6 = -3\ 0$ $g = 2$

$b_3 = 0\ 5$ $b_7 = 0\ 5$

$b_4 = 1\ 0$

and x_i $i = 1\ 2$ 7 t ke $+1$ or -1 values

tage 1 $F_1(x_1 \ x_2 \qquad x_7) = U_1(x_1 \ x_2 \ x_3) + V_1(x_2 \ x_3 \qquad x_7)$

$$U_1(x_1 \ x_2 \ x_3) = -2x_1 b_1 + 2x_1 \sum_{i=2}^{3} x_i a_{|i-1|} + x_1^2 a_0$$

$$= -3x_1 + 2x_1[0\ 5\ x_2 - 0\ 25 x_3] + x_1^2$$

T ble 3 3 is obt ined by determining th value of x_1 that minimizes U_1 for a fixed x_2 and x_3

Table 3 3   Stage 1

x_2	x_3	x_1^*	U_1^*
-1	1	1	-5/2
-1	1	1	-3 5
1	1	1	-0 5
1	1	1	-1 5

Since $x_1 = 1$ for all th values of x_2 x_3 we can decide at this stage itself that the first variabl is $x_1^* = 1$

Stag 2 $U_2(x_2 x_3 x_4) = -2x_2 b_2 + 2x_2 \sum_{\iota=3}^{4} x_\iota a_{|2 \iota|} + x_2^2 a_0 + U_1(x_2 x_3)$

As the calculations arc dentical in all the stages we have presented th f nal results in Tabl s J 4-3 8 corresponding to stages 2 to 6

Table 3 4 Stage 2

x_3	x_4	x_2^k	U_2
-1	-1	1	-4
1	1	1	-5
1	-1	1	-3
1	1	1	-4

Therefore the second variable $x_2^* = 1$

Table 3 5 Stage 3

x_4	x_5	x_3	U_3
-1	-1	1	-3 5
-1	1	1	-4 5
1	-1	1	-4 5
1	1	-1	-J J

Table 3 6 Stg 4

x_5	x_6	x_4^k	U_4^k
-1	-1	1	-6
-1	1	1	-7
1	-1	1	-3
1	1	1	-4

Ta le 3 7 Sta e 5 | | | | Table 3 8 Stage 6

x_6	x_7	x_5^*	U_5	x_6	x_7	U_6
-1	-1	1	-7	1	-1	9 5
-1	1	-1	-6 5	1	1	-12 5
1	-1	-1	10 5	1	-1	-2 5
1	1	1	9 5	1	1	-1 5

From Taol 3 8 we   th t x_6 = 1 and x_7 = 1 On back ubstitution Tao l   7 gives x_5 = 1 It may ls b not a hat Table 3 6 na labl 3 7 or stage 4 and 5 r spectively directly indicate hat $x_4^* = 1$ and $y^t = -1$ Ising the values of x_4^* and   in Table 3   we obta n $x_3 = -1$

VITERBI LGOPITHM FOR MINIMIZING QF

For the purpos of co par son we hav given a bri f des cription of VA in this section

R call that the SfQF min mization p oblem arose in conn ction with tran is ion on digit l cata through a finite impulse respons han e modelled as finite emo y transversal filter

At the kth instar t he outout of su h ilt r can be repr sented in terms   ou x_k x_{k-1} x_{k-g} as

$$y_k = \sum_{l=0}^{q} h_l x_{k-l} \qquad k - 1\,2\,3$$

wh r g is th memory o f transvcr al filter   Thus y_k 1 a
functio1 of the pı sent ınput as well a the prevıous state of
h trarsversal filtcr   Lct S_k denotc the tatc of the trans-
versal filt r at th kth ıı tan   Th ı

$$S_k = (x_k\ x_{1-1} \qquad x_{k\ g+1}) \qquad x_k = 0 \ if\ k < 0$$

I1 t rms of u_k s y_l can be expresscd a

$$y_k = f(h_o\ h_1 \qquad h_g\ S_k\ S_{k-1})$$

Th ıcfoıc given ın ınpu scquencc $\{x_k\}$ we have a correspond-
ıng st t sequcnce $\{S_k\}$ anc an output scquence $\{y_k\}$ This
co respondcnce ıs one to onc   This observatıon ı crucıal
to the dcvelopmont of VA   Due to this corr spondence we can
mınımıze $F(\underline{x})$ ov r S_k s instead of x_k s   For this purrose
decompo c $F(\underline{x})$ as

$$F(x) = f_1(\underline{x}) = [-2 \sum_{l=2}^{n} x_l b_l + \sum_{l=2}^{m} \sum_{j=2}^{n} x_l x_j a_{|l-j|}] +$$

$$[-2x_1 b_1 + 2x_1(x_2 a_1 + x_3 a_2 + \qquad x_{g+1} a_g) + x_1^2 a_o]$$

Note that the quantıty in ıde the second paı r of bıackets ıs
only ı function of $x_1\ x_2 \qquad x_{g+1}$ Le

$$f_1(\underline{x}) = u_1(x_1 \qquad x_{g+1}) + v_1(x_2 \qquad x_m)$$

It is easy to see that $u_1(x_1 \ x_2 \qquad x_{g+1}) = u_1(S_{g+1} \ S_g)$

Now v_1 can be similarly decomposed a

$$v_1(x_2 \ x_3 \qquad x_m) = v_2(x_3 \ x_4 \qquad x_m) + u_2(S_{g+2} \ S_{g+1})$$

In a similar fashion we can write for the kth stage

$$v_k(x_{k+1} \qquad x_m) = v_{k+1}(x_{k+1} \qquad x_m) \ u_k(S_{g+k} \ S_{g+k-1})$$

$$k = 1 \ 2 \ 3 \qquad m \ g$$

Therefore $F(\underline{x})$ can be written as

$$F(S_1 \ S_2 \qquad S_m) = \sum_{k=1}^{m-g} u_k(S_{g+k} \ S_{g+k-1}) + v_{m \ g}(S_m)$$

where

$$u_1(\ _{g+k} \ _{g+k-1}) = -2x_k b_k + 2x_k \sum_{1=k+1}^{g+k} x_1 a_{|k-1|} + x_k^2 a_0$$

$$k = 1 \ 2 \ 3 \qquad m-g$$

We wish to note that the derivation of the foregoing recursive equations is based on the approach given by Hayes [41] The optimality of the algorithm follows from Lemma 3 1

The algorithm works as follows In the first stage of the algorithm $u_1(S_{g+1} \ S_g)$ is minimized with respect to S_g for fixed S_{g+1} A S_{g+1} can assume anyone of the q^g states we have to store u_1^t value corresponding to q^g states As S_g and S_{g+1} differ in only one place d termination of optimum S_g given S_{g+1} is effectively one variable optimization To complete our discussion let

$$\Gamma(S_g \quad S_m) = f_1(S_g \quad S_1) = u_1(S_{g+1} \ S_g) +$$

$$\sum_{k=2}^{n-g} u_1(S_{g+k} \ S_{g+k-1}) + v_{m-g}(S)$$

Hence

$$\min_{S_g} \ \min_{S_m} f_1 = \min_{S_{g+1}} \ \min_{m} \left[\sum_{k=2}^{n-a} u_k(S_{g+k} \ S_{g+k-1}) + v_{m-g}(S_m) + \right.$$

$$\left. \min_{S_g} u_1(S_{g+1} \ S_g) \right]$$

Define $u_k^{\prime} = u_k(S_{g+1} \ S_{g+1-1}) \qquad u_o^{\ast} = 0$

Th n th recur$1 v$e e uations for /A an b written as

$$u_k^{\ast} = \min_{S_{g+k-1}} \left[u_k(S_{g+1} \ S_{g+k-1}) \ u_{k 1} \right] \quad k - 1 \ 2 \qquad m-g$$

Finally the optimum value of F F 1 g ven as

$$F^{\ast} = \min_{S_m} \left[u_{m-g} + v_{m g}(S_m) \right]$$

Result obtained by applying VA to E ample 3 1 is snown in he F_gure 3 1 In t e ficure w° have shown the survivor paths hrough the trelli In h figure dotted lines indicate alter-nate survivors giving same function values From the figure it nay be noted that (1 1) 1 node common to all the survived paths in stage 3 Th s nplies that $\lambda_1 = 1$ Thus decisions regarding data can be mac ven before completing the forward recursion process This is due to the merging of the urvivor o ths through the Trellis
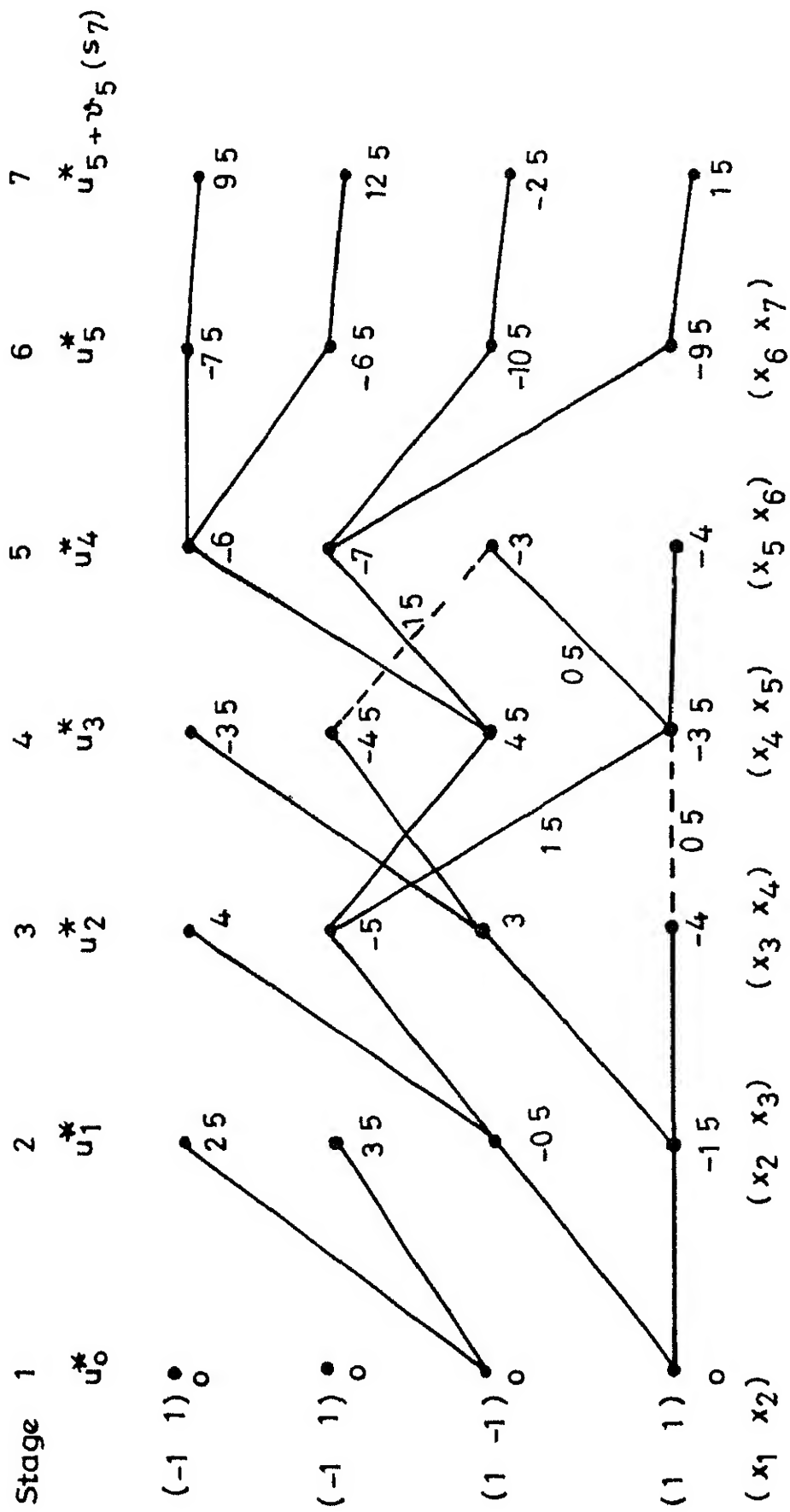
FIG 3 1 Trellis diagram showing survivor paths

Let us now consider the solution obtained by applying
ML-RDA to Example 3 1   From the Tables 3 3-3 8 w can easily
see that $x_1 = 1$ Similarly table given for stage 2 gives
$x_2^* = 1$ The third stage shows that $x_3 = -x_4^*$ Fourth stage
gives $x_4^* = 1$ and fifth stage $x_5 = -1$ These aspect do not
become clear from the Trellis diagram shown in Figure 3 1

3 4   TRATECIES FOR REDUCING COMPUTATIO AL AND STORAGE
   REQUIREMENTS

Throughout the discussion of th recursiv deconvolution
algorithm we have used tables to how the results obtained at
each stage of the algorithm   In actual practice su h tables
are not necessary   Notice that the table used in the first
stage is used only by the second stage   Therefore in the
second stage we can use the updated v rsion of the first table
For the purpose of illustratior w have shown the development of
the tables for Example 3 1 in Figure 3 2   Back substitution
gives the path snowing dotted lines in he figure It is obtain
ed as follows x_6 and x_7^* ar decided by noting that U_6^* is
min mum at that point x_5 for the state $(x_6^*$ $x_7) = (1$ 1) is
-1 x_4^* for the state $(x_5^*$ $x_6) = (-1$ 1) is 1 x_3 for the state
$(x_4$ $x_5) = (1$ -1) is -1 x_2^* and x_1^* are obta ned similarly

As the procedure is same for a general n-variable BTQF we
see that the storage requirem nt is q^g registers of length (m+1)
ach for storing x_1^* s at each stage and the accumulated minimum

Stage 1

x_1	x_2	y_3	U_1^t
1	-1	-1	-5/2
1	-1	1	-3 5
1	1	-1	-0 5
1	1	1	-1 5

Stage 2

x_1^*	x_2	x_3	x_4	U_2
1	1	-1	-1	-4
1	1	-1	1	-5
1	1	1	-1	-3
1	1	1	1	-4

Stage 3

x_1^*	x_2^*	x_3^*	x_4	x_5	U_3^*
1	1	1	1	-1	-3 5
1	1	1	-1	1	-4 5
1	1	1	1	-1	-4 5
1	1	-1	1	1	-3 5

Stag 6

x_1^*	x_2	x_3^*	x_4^t	x_5^*	x_6	x_7	U_6^t	State	
1	1	1	1	-1	-1	-1	-9 5	-1	-1
1	1	1	1	-1	-1	1	-12 5	-1	1
1	1	-1	1	-1	1	-1	-2 ɔ	1	-1
1	1	-1	1	-1	1	1	-1 5	1	1

Figure 3 2   Developm nt of the tables in ML-RDA for Example 3 1

value at every stag   Thus the storage requirement grows
linearly with the block length   Further since at every stage
e need q^g function evaluations the computational complexity
of the lgorithm grows linearly with block length and exponen-
tially with channel me ory

Even with the foregoing strategy for storing the tables
the alg rithm still re ures a large amount of storage and
computations for even mod rate block lengths   Thus if th
recursive algorithm is to b come acceptable it must hav
additional features for minimising storage and computational
require ments   Towards this end w now propose and examine a
method for reducing the complexity of ML-RDA which is expected
to re ult in a greater savings in storag and computations

To b gin with let us examine Tables 3 3-3 8 for Exampl 3 1
For instance consider Table 3 5 obtained at the 3rd stage
Note that if we are able to xpress x_3^* as a function of x_4 and
x_j substi uting for x_3^* in the objective function eliminates
the need for the table in th n xt stage computations Moreover
for the purpo e of b ck substitu ion we need retain only the
functional form of x_3 In this case it is easy to see that
$$x_3^* = -x_4$$

If at every stage we are able to obtain such simple
r present tion stor ge requir ments are reduced to a minimum
In the example under consideration all the functions have simple
representations as given below

tag	1	2	3	4	5	6

Functions $x_1=1$ $y_2=1$ $x_3^*=-x_4$ $x_4=1$ $x_5^*=-1$ $x_6=1$ and $x_7=1$

such simple functional forms cannot always be guaranteed For the time being assume that such representations are possible Then the next question would be can we arrive at the functional form without numerating all possible values of the fixed variables? If this is possible we would also have achieved reduction in the computational complexity

However in general this appears to be a difficult problem In the first place functional form of x_1 may be complicated due to which storage of such function as well as substitution into the objective function may be difficult Secondly it may be difficult to directly obtain the functional forms Therefore to begin with instead of considering the most general case we study the binary case in the next section

4 1 Binary Data Transmission

When the data symbols assume one of the two values 1 or -1 the objective function can be converted into another function called pseudo Boolean function (PBF) [57] which is a function of Boolean variables with real coefficients For this we use the transformation $y = (1+x)/2$ (which means y takes 0 and 1 as x takes -1 and 1 respectively) With the help of this transformation the quadratic function $F(\underline{x})$ can be converted into a PBF $F(\underline{y})$ where y_1 y_2 y_m are Boolean variables

Le $F(\underline{x})$ expressed in the matrix vector form as

$$F(\underline{x}) = -2x^T b + x^T A \underline{x} + \underline{r}^1 \underline{r}$$

Je use th transformation $\underline{y} = (\underline{1}+x)/2$

where $\underline{1}$ is a vector of all ones With this t an for nation we obtain

$$\Gamma(\underline{y}) = -2\underline{y}^T + \underline{y}^T A \underline{y} + p$$

where $b = 2(\underline{b}+A\underline{1})$ $A = 4A$ $p = 21^T b + \underline{1}^1 A\underline{1} + \underline{r}^T r$

The con tant term p can be ignored for he purpose of minimizing $F(\underline{y})$

$\Gamma(\underline{y})$ is sa me as $\Gamma(\underline{x})$ Therefore the IL-PDA given in S ction 3 2 applies to this case with the difference that the tables at each tage now contains Boolean variables As our interest is in xpressing x_1^* in terms of x_{1+1} x_{1+2} x_{1+g} at the ith stage we observ that the table in this case is a truth table for x_1^t in terms of x_{1+1} x_{1+2} x_{1+g} Now using re ults from the theo y of Boolean functions [72] we can repre-sent x_1 in some convenient form For ins ance minimal sum of produ t or product o sums form

Thus we have shown that in the case of binary data there exists a procedure for expre sirg x_1^+ in t ms of x_{1+1} x_{1+2} x_{1+g} in the ith stage Howe er simple expressions for x_1^* may not always be obtained Let us consider the worst case where x_1^*

is rcor sented in terms of x_{i+1} x_i g The truth tabl
or x_i^* will contain 2^g ntries In this case h stor ge can
bc reduced to some extent by obta_ning a mi imal representa
tion of x_i i thods such as Quin?-McClausky or any such
similar m thod [72] could oe used for this puipose However
ic may fave to use a special purpo c compute~ vhich imolcment
Boolean function minimization algor thm

Now vc cons der the probl m of obtaining unctional repre-
senta ion for y_i^* withou enum ating all po ible 2^α cases by
using an aljorithm d vcloped by Hammer et al [57] we show that
the computational conplcx ty can bc reduced nt vcry stage
without sacrifying optimality Towards this end we use
L mmn 3 1 to derive a r cursive algorithm dcvcloped originally
b/ Hammer et al [o7] for minimizing nonlinea PBF s in the
ncyt section The approach taken herc i much simoler compared
to the derivation given in [o7]

3 4 2 Minimization of Pseudo-Boole n Functions

Consider a PBF $F_1(x_1 \, x_2 \quad x_m)$ of m variables Since
pseudo-Boolean function is linear in cach of its variables wc
can write

$$F_1(x_1 \, x_2 \qquad x_m) = x_1 U_1(x_2 \, y_3 \qquad x_{11}) +$$
$$V_1(x_n \, x_3 \qquad x_m)$$

Application of Lemma 3 1 gives

$$\min_{x_1 \quad x_m} F_1 = \min_{x_1 \quad x_m} [V_1(x_2 \; x_3 \qquad x_m) +$$

$$\min_{y_1} x_1 U_1(x_2 \; x_3 \qquad x_m)]$$

Now since x_1 s are Boolean variables it follows that

$$x_1 = \begin{cases} 0 & \text{if} \quad U_1(x_2 \; x_3 \qquad x_m) > 0 \\ \emptyset & \text{if} \quad U_1(x_2 \; x_3 \qquad _m) = 0 \\ 1 & \text{if} \quad U_1(x_2 \; x_\cup \qquad x_m) < 0 \end{cases}$$

where $\emptyset$ is a don t care variable (x_1 arbitrary)

Therefor a Boolean function representation ror x_1 can be obtained by determining those $x_2 \; x_3$ x_m for which $U_1 < 0$ substitution of x_1^* into F_1 gives F_2 The other stages of the minimization follow the same pattern as in the case of recursive procedure described earlier Thus at the ith stage we have

$$\min_{x_1 \quad x_m} F_1 = \min_{x_{1+1} \quad x_m} [V_1(x_{1+1} \; x_{1+2} \qquad x_m) +$$

$$\min_{y_1} x_1 U_1(x_{1+1} \; x_{1+2} \qquad y_m)]$$

and

$$y_1^* = \begin{cases} 0 & \text{if} \quad U_1 > 0 \\ \emptyset & \text{i} \quad U_1 = 0 \\ 1 & \text{ir} \quad U_1 < 0 \end{cases}$$

As we are interested in only one solution ϕ can be fixed as O or 1 and we can solve $U_1 \leq O$ or obtaining x_1 or $U_1 \geq O$ for obtaining x_1^*. Hammer et al [57] have also developed an algorithm or determining all solutions which satisfy $U_1 \leq O$. This algorithm can be used to reduce the computational comple-t/ a each stage

From the foregoing discussion it is clear that in the binar/ case it is possible to r duce storage and computational comolexit/ of the RDA if fast processors are available for minimization of Boolean functions and for implementing Hammer t al procedure for obt ining y_1

It may be noted that determination f uni ial representa-ion of Boolean functions is a c'as ical proolem in the theory of switching functions. Although a large number of methods are available [72] no definite claim can be nad in favour of one approach over the other

In summary the over ll effici ncy of the RDA if impl - n nted in the manner described in this ection depends to a great extent on the availability of fast algorithms for obtaining minim l repres rtations for x_1^* either in the Boolean func ion form or in the pseudo-Boolean function form

In the nonbinary case however no such claims can be made. This aspect needs furth r investigation

3 5 P_CURSIVE ALGO₁ITHM FOR D⁻CODING BLOCK ERROR COPRECTING CODES

As an application of the recursive algorithm developed in the previous section we consider maximum likelihood decoding of binary block codes Our approach consists in formulating the ML decoding prollem as an equivalnt probler of minimizing a pseudo-Boolean funcion of several variables It may be noted that the algorithm de/ loped in this section can also be used for maximum likelihood accocing of convolution cod s

We wish to note hat Volf [73] has studied the application of V/ to decode blocl coces by con tructing Trellis diagrams The approach consid r d here doe not require Trellis construction and it seems to provide a rore general method for maximum likelihood decoding of binary blocl codes

The codes considered here are over GF(2) where GF(2) denotes Galois field with two elements O and 1 [72] For such codes the ML decoding is quivalent to solving th set of linear equations $G\underline{x} = r$ over GF(2) such that the Hamming distance between $G\underline{x}$ and r is minimum when all the code words are assumed to equiprobable The matrix G is called the generator matrix of the code in the parlance of coding theory [6] [7] The received vector $\underline{r}$ is an additive combination of the transmitted codevector which is not observable at the receiver and the noise vector $\underline{w}$ That is $\underline{r} = \underline{c} \oplus \underline{w}$ where $\oplus$ denotes addition

operation in GF(2) It may be noted that the lements of $\underline{r}$
and $\underline{w}$ are e ther 0 or 1 We also assume that $\cap$ is an nxm
m trix

The Hamming distance between two vectors $\underline{x}$ and $\underline{y}$ over
GF(2) is defined as the number of nonzero entri s in x $\oplus$ $\underline{y}$
It is also known as the weight of $\underline{x}$ $\oplus$ y d noted as $w(\underline{x}$ $\oplus$ $\underline{y})$

Since GF(2) elements are also elements of the real field
we can show that

$$\underline{p} \oplus \underline{q} = \underline{p} + \underline{q} - 2\underline{p}\ \underline{q}$$

where p q denote pointwise multiplication Ne now make use of
the property that if y s 0 1 real vector then

$$w(\underline{y}) = \underline{1}^T \underline{y}$$

where $\underline{1}$ is column vector of all ones Therefore it is easy
to see that

$$w(\underline{p}\ \oplus\ \underline{q}) = w(p+q-2\underline{p}\ q) = w(p)+w(q)\ 2w(p\ \underline{q})$$

Thus maximum likelihood decoding problem can be formulated as

$$\min_{x}(w(G\underline{x})+w(\underline{r})-2w(G\underline{x}\ \underline{r}))$$

$$= \min_{x}(\underline{1}^T G\underline{x}+\underline{1}^T\underline{r}-2\underline{1}^T(G\underline{x}\ \underline{r})) \tag{3 5}$$

Note that $G\underline{x}$ is to be computed ov r GF(2)   Therefore
we n ed a suitable representation for th elements of $G\underline{x}$ Let
$\underline{a}_1$ a_2 $\underline{a}_n$ denote n rows of G   Then we can write $G\underline{x}$ as
$(a_1\underline{x}$ $a_2\underline{x}$ $a_n\underline{x})$ wh re $\underline{a}_1\underline{x} = a_{11} x_j \ominus a_{12} y_2 \oplus$ $\oplus a_{1m} x_m$
$\parallel$ c n us the r lation

$$u \oplus v = u + v - 2uv \qquad (3\ 6)$$

in ord r to convert $\underline{a}_1\underline{x}$ into n equival nt ps udo-Boolean
function   However   uch a representation becomes too compli
cated wh n more variables re involved   A more compact form
of representation is n c ss r) to obt 1 a representation which
reflects the gen rator matrix explicitly   Towards that end
consider the transform tion $u = 1 - 2u$ and $v = 1 - 2v$ Substi-
tuting for u and v on th right hand side of (3 6) w obtain
$u \ominus v = (1 - u\ v)/2$ which   is a more compact expression
compared to $u + v - 2uv$ We can furth r show that

$$u \oplus v \oplus w = (1 - u\ v\ w)/2 \quad \text{wh re} \quad w = 1 - 2w$$

In g neral w have $\sum_{1=1}^{m} c_1 x_1 = (1 - \prod_{1=1}^{m} z_1^{c_1})/2$ where   denotes

summation in GF(2)   and $c_1 \in GF(2)$ $1 = 1\ 2$ m
and $z_1 = 1 - 2x_i$ Now for $\underline{x}$ we have

$$\underline{a}_1\underline{x} = \frac{1}{2} (1 - \prod_{j=1}^{m} z_j^{a_{1j}}) \quad 1 = 1\ 2 \qquad m$$

$$z_j^{a_{1j}} = \begin{cases} 1 & \text{if } a_{1j} = 0 \\ z_j & \text{if } a_{1j} = 1 \end{cases}$$

Note that the representation for $\underline{a}_1\underline{x}$ clearly shows the

ffect of a_{1J} s

going oacl to the minimization problem given by

(3 5) n can ignore $\underline{1}^T\underline{r}$ for th purpos of minimization as it is

a const nt The rest of the terms can be combined as

$$\min_{x} (\underline{1}^T G\underline{x}\ \underline{r}\ )\quad \text{wh rc}\quad \underline{r}\ = \underline{1}\text{-}2\underline{r}$$

Let $\underline{d} = (d_1\ d_2 \qquad d_n)^T$ where $d_J = \prod_{1=1}^{m} z_1^{a_{1J}}\quad J = 1\ 2 \qquad n$

Then $G\underline{x} = \frac{1}{2}(1\text{-}\underline{d})$ Th refore $G\underline{x}\ r\ = \frac{1}{2}\underline{1}\ r\ - \frac{1}{2}\underline{d}\ \underline{r}$

Thus we have to find

$$\min_{\underline{z}} (-\frac{1}{2}\underline{1}^T(\underline{d}\ r\ )) = \min_{\underline{z}} (-\frac{1}{2}\sum_{J=1}^{n} d_J r_J)\quad \text{where } r_J\ = 1\text{-}2r_J$$

With the objective function in this form we can now

proceed with the r cursive optimization algorithm iscussed in

Section 3 4 1 To round up the for going aiscu sion we give

thre illustr tive examples

Example 3 2 (Hamming Code) Consider the generator matrix G

given by

$$G^T = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Let $\underline{x} = (1\ 1\ 1\ 0)^T$ be the transmitted vector   Then

$\underline{c} = (1\ 1\ 1\ 0\ 0\ 0\ 0)^T$ is th transmittcd code v ctor

L t $\underline{r} = (0\ 1\ 1\ 0\ 0\ 0\ 0)^T$ be the received vector   For this situa ion we have

$$d_1 = z_1 \quad d_2 = z_2 \quad d_3 = z_3 \quad d_4 = z_4 \quad d_5 = z_1 z_2 z_4$$

$$d_6 = z_1 z_3 z_4 \quad \text{and} \quad a_7 = z_2 z_3 z_4$$

Therefor the fuıction to be minimized is given by

$$f(\underline{z}) = -\frac{1}{2}(z_1 - z_2 - z_3 + z_4 - z_1 z_2 z_4 + z_1 z_3 z_4 + z_2 z_3 z_4)$$

Negl cting $1/2$ we can write $f(\underline{z})$ as

$$f(z) = z_1(-1 - z_2 z_4 - z_3 z_4) - z_2 - z_3 + z_4 + z_2 z_3 z_4 =$$

$$U_1(z_1\ z_2\ z_3\ z_4) + V_1(z_2\ z_3\ z_4)$$

Minimization of U_1 with r spect to z_1 for fixed z_2 z_3 and z_4 giv s the following table for stage 1

Stage 1						Stage 2			
z_2	z_3	z_4	z_1^*	U_1^*		z_3	z_4	z_2^*	U_2^*
-1	-1	-1	1	-3		-1	-1	-1	-3
-1	-1	1	1	-1		-1	1	1	-3
-1	1	-1	1	-1		1	-1	-1	-3
-1	1	1	1	-1		1	1	1	-3
1	-1	-1	1	-1					
1	1	1	1	-1					
1	1	-1	-1	-1					
1	1	1	1	-3					

In terms of U_1^* we can write $f(z)$ as $f(\underline{z}) = U_1^*(z_2 \; z_3 \; z_4) + z_2(1-z_3 z_4) + z_3 - z_4$

Minimization of $f(\underline{z})$ with resp ct to z_2 gives the table for stage 2 Note th t U_2^* is independent of z_3 and z_4 Therefore $f(z) = -3 + z_3 - z_4$ Minimizing finally with respect to z_3 and z_4 we obtain

$f(\underline{z}) = -3$ when $z_3^* = -1$ and $z_4 = 1$ which on back substitution gives $z_2^* = -1$ and $z_1 - -1$ Therefore we obtain (1 1 1 0) as the recovered data vector

Example 3 3 (Reed Muller Cod)   Consider the generator matrix given by

$$G^T = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

Let $\underline{x} = (1\ 1\ 1\ 1)^T$ be the transmitted vector

$\underline{c} = (1\ 0\ 0\ 1\ 0^*\ 1\ 1\ 0^*)$ and $\underline{r} = (1\ 0\ 0\ 1\ 1\ 1\ 0\ 0)^T$

In this case the function to be minimized is given by

$f(\underline{z}) = -\frac{1}{2} [-z_1 + z_1 z_2 + z_1 z_3 - z_1 z_2 z_3 - z_1 z_4 - z_1 z_2 z_4 + z_1 z_3 z_4 + z_1 z_2 z_3 z_4]$

$f(\underline{z})$ can b written as

$f(\underline{z}) = \frac{z_2}{2} (-z_1 + z_1 z_3 + z_1 z_4 - z_1 z_3 z_4) + \frac{1}{2} (z_1 - z_1 z_3 + z_1 z_4 \; z_1 z_3 z_4)$

For this case the table obtained for the first stage is shown below

tage 1

z_1	z_3	z_4	z_2^*	U_2
1	-1	-1	-1	-2
-1	-1	1	$\emptyset$	0
-1	1	-1	1	-1
-1	1	1	$\emptyset$	0
1	-1	-1	1	-2
1	-1	1	$\emptyset$	0
1	1	-1	$\emptyset$	0
1	1	1	$\emptyset$	0

$$U_2^*(z_1\ z_3\ z_4) = \min_{z_2} \frac{z_2}{2}\ (-z_1+z_1z_3+ z_1z_4-z_1z_3z_4)$$

Note that in the first stage we have considered z_2 instead of z_1 z_1 also can be consid red as the f rst stage variable

From the tabl it is not difficult to s e that $z_2^t = z_1z_3z_4$ Substituting z_2^* in $f(\underline{z})$ we obtain

$$f_2(z_1\ z_3\ z_4) = \frac{z_1}{2}\ (1-z_3+z_4-z_3z_4) + \frac{1}{2}\ (-z_3z_4+z_4+z_3\ 1)$$

Tabl for th sccond stag minimization is given below

Stage 2

z_3	z_4	z_1^t	U_1
-1	-1	$\emptyset$	0
-1	1	-1	-2
1	-1	$\emptyset$	0
1	1	$\emptyset$	0

From the table it is easy to see that $z_1 = z_3$ Substituting

for z_1 in $f_2(z_1 \; z_3 \; z_4)$ we obtain

$$f_3(z_3 \; z_4) = z_3 - 1$$

Therefore we obtain $z_3 = 1$ $z_4^* = \emptyset$ $z_2^t = \emptyset$ $z_1^* = -1$

where $\emptyset$ d notes either 1 or -1   Thus the maximum likelihood

solutions are

$$(-1 \; 1 \; -1 \; 1) \quad \text{and} \quad (-1 \; -1 \; -1 \; 1)$$

The corresponding 0 1 vectors are $(1 \; 0 \; 1 \; 0)$ and $(1 \; 1 \; 1 \; 1)$

Thus although the transmitted vector is $(1 \; 1 \; 1 \; 1)$ we obtain

$(1 \; 0 \; 1 \; 0)$ also as a solu ion   This is because two errors were

introduced in the code vector at the places indicated by *

in $\underline{c}$ The same received vector $\underline{r}$ can be obtained by introducing

two rrors in the locations 2 and 4 of the code vector $\underline{c}_1$ obtained by $(1\ 0\ 1\ 0)$ as shown below

$$\underline{c}_1 = (1\ 1\ 0\ 0\ 1\ 1\ 0\ 0)^T$$

Example 3 4   Con ider the generator matrix given by

$$G^T = \begin{bmatrix} 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

d_j s can be obtained by reading columnwise in G^T and associating z_i s whenever 1 s are encountered   For instance w obtain d_{16} by noting that the last column contains 1 in the z_6th position   Therefore $d_{16} = z_6$ The objective fun-ction is then given by

$$f(z_1\ z_2 \quad z_6) = -\frac{1}{2} \sum_{i=1}^{16} d_i r_i = -\frac{1}{2} (-z_2 - z_1 z_2 + z_1 z_3 + z_1 z_2 z_3 -$$

$$z_2 z_4 - z_2 z_4 - z_3 z_5\ z_3 z_4 z_5 - z_4 z_6 - z_4 z_5 z_6 - z_5 -$$

$$z_5 z_6 + 2 z_6)$$

where we assume that the transmitted vector is $(0\ 1\ 0\ 0\ 1\ 0)^T$

$$c = (0^* \ 0\ 1\ 1\ \cup\ 1^* \ 1\ 1\ 1\ 1^* \ 0\ 1\ 1\ 1\ 0\ 0)^T$$

and $\underline{r} = (1\ 0\ 1\ 1\ 0\ 0\ 1\ 1\ 1\ 0\ 0,1\ 1\ 1\ 0\ 0)$

Note that 3 errors occur in this case

Now carrying out the steps of the recursive algorithm we obtain th following tables

Stage 1   Stage 2

z_2	z_3	z_1	U_1^*
1	1	1	1
1	-1	-1	-3
-1	1	1	1
-1	-1	1	-1

z_3	z_4	z_2	U_2^k
1	1	1	-4
1	-1	1	-2
-1	1	$\emptyset$	-2
-1	-1	1	-2

$$U_1 = z_1(z_2 - z_3 - z_2 z_3)$$

$$U_2 = z_2(1 + z_4 + z_3 z_4) + U_1^k$$

In stage 2 it is easy to note that $z_2^t = -z_3 z_4$ if we put $\emptyset = 1$

Stage 3   Stage 4

z_4	z_5	z_3^*	U_3^*
1	1	-1	-4
1	-1	1	-6
-1	1	$\emptyset_1$	-2
-1	-1	$\emptyset_2$	2

z_5	z_6	z_4^*	U_4^*
1	1	-1	-4
1	-1	1	-6
-1	1	1	-6
-1	-1	1	-6

$$U_3 = z_3(z_5 + z_4 z_5) + U_2$$

$$U_4 = {}_4({}_6 + z_5 z_6) + U_3^k$$

69

$$z_3^* = -z_4 z_5, \quad \text{if} \quad \emptyset_1 = 1 \quad \text{and} \quad \emptyset_2 = -1$$

Stage 5

z_6	z_5	U_5
1	-1	-8
-1	$\emptyset$	6

$$U_6 = -2z_6 + U_5^*$$
$$U_6^* = -10$$
$$z_6^* = 1$$

$$U_5 = z_5(1+z_6) + U_4$$

Tracing back through the tables we obtain

$$z_6 = 1 \quad z_5 = -1 \quad z = 1 \quad z_3 = 1 \quad z_2 = -1 \quad z_1^* = 1$$

Therefore th maximum likelihood solution vector is

$(0\ 1\ 0\ 0\ 1\ 0)^T$ which is the transmitted data vector

CHAPTEP   4

DECISION FEEDBACK DECONVOLUTION SCHEMES

In the previous chapter we derived a ML RDA whose
computational complexity and stor ge requirements grow linearly
with the block length and exponentially with th channel memory
We showed that attempts to further alleviate computational ard
storage requirements in th binary ca e lead to minimization
of pseudo Boolean func ons which in general is a rather
difficult problem   The computational and storage requirements
how ver can be reduced if one is willing to accept suboptimal
 ch mes Towards this end in this chapter we con ider the
possibility of incorporating decision ecdback idea for
recov ring data in BD systems   We propose and analyse four
cecision feedback data recovery (DF-DR) schemes   All these
s emes are based centrally on the idea of nonlinear Gaussian
elimination method described below

Consider the received vector r given by (2 7)

$$\underline{r} = H\underline{x} + w \tag{4 1}$$

Under the assumption of r al valued data and white Gaussian
noise the ML data vector is obtained by solving

$$H^T H x = H^T r \tag{4 2}$$

One method of solving (4 2) is to use the Gaussian elimina-
tion (GE) algorithm which consists of forward elimination proce-
dure (in which $H^T H$ is reduced to an upper triangular matrix) and
back substitution (in which the solution is obtained by recursi
vely solving th resulting set of equations) Now we can incor
porate the discrete nature o data symbols into GE algorithm in
the follo ing n nner While back substitution instead of using
the calculated valu s directly we can us the value obtained by
passing the calculated valu th ough a nonlinear d cision
device which assigns an appropriate valu from the finite
alphabet Since this proc dure involve in roduction of a non
linear device between t e fo ward elimination and back substi-
tu ion stages the method s called nonlin ar Gaussian elimina-
ion

This chapter is organized as follows In S ction 4 1 we
us Cnolesky decomposition t chnique to derive a DF-DR scheme
which does not t   noise variance into account   section 4 2
gives a DF-DR sch ne which minimizes moar s uare error between
the actual data and its estimat A numerically stable DF DR
sch me is derived in Section 4 3 This cl ene is based on the
minimum property of Fourier expansion in inite dimensional
Hilbert spaces In Section 4 4 a DF-DR sch m is derived by
modifying Austin s [13] assump ions regarding decision feedback
equalizer for CSDT systems Ir S ction 4 5 ic show that the
modified Austin s pproach is also essen ial'y equivalent to a

nonlinear Gaussian elimination method. In S ction 4 6 we list some observa ions on the methodology used in this chapt r for motivating the idea of decision feedback

4 1 DECI ION FEEDBACK SCHEME USING CHOLESKY DECOMPOSITION

Cholesky cecompo ition technique [74] deals with decomposition o positive definite matric s into product of nonsingular lover and upper triangulai matrices. If V denotes a lower triangular matrix we can write $H^T H$ as

$$H^T H = V^T V \qquad (4\ 3)$$

Using (4 3) (4 2) can be written as

$$V^T \underline{x} = V^{-1} H^T \underline{r}$$

This is equivalent to the forward elimina ion step of GE algo- r thm. Since V^T is an upper triangular matrix estimate of the last element of $\underline{x}$ is obtained as

$$\tilde{x}_m = c_m / v_{mm}$$

where c_m is the mth elemer t of $c = V^{-1} H^T \underline{r}$ and v_{mm} is the 1 h diagonal el ment of V^T. Now let $\hat{x}_m$ be the value obtained by passing $\tilde{x}_m$ through a decision device. We use $\hat{x}_m$ instead of $\tilde{x}_m$ for calculating $\tilde{x}_{m-1}$ given by

$$\tilde{x}_{m-1} = (c_{m-1} \quad v_{m\ 1\ m}\hat{x}_m)/v_{m-1\ m-1}$$

Now $\hat{x}_{m-1}$ will be the recovered data decided on the basis of $\tilde{x}_{n-1}$. Using m and $\hat{x}_{m-1}$ $\tilde{x}_{n-2}$ is calculated. This procedure is continued till $\hat{x}_1$ is obtained.

In the foregoing approach last data of the block is recovered first whereas in the case of decision feedback data recovery schemes for CSDT first data is recovered first. Although which data value is recovered first is immaterial in the case of BDT, for the sake of uniformity we incorporate this feature of CSDT systems in the following manner.

$H^T H$ can also be written as a product of upper and lower triangular matrices. That is

$$H^T H = U^T U \qquad (4\ 4)$$

where U is the upper triangular matrix. One method of obtaining U is as follows.

Let $A_m = H^T H$ and $A_{m-1} = U_{m-1} U_{m-1}^T$

Then A_m and U_m can be written as

$$A_m = \begin{bmatrix} a_{mm} & d_{n-1}^T \\ & \\ c_{m-1} & A_{n-1} \end{bmatrix} \quad \text{and} \quad U_n = \begin{bmatrix} y & e_{m-1}^T \\ & \\ 0 & U_{m-1} \end{bmatrix}$$

Then

$$A_m = \begin{bmatrix} y^2 + e_{m-1}^T e_{n-1} & e_{m-1}^T U_{m-1}^T \\ & \\ U_{m-1} e_{n-1} & U_{n-1} U_{n-1}^T \end{bmatrix}$$

Therefore $y = (a_{mm} - \underline{e}_{m-1}^T \underline{e}_{m\ 1})^{1/2}$ and $\underline{d}_{m-1} - U_{m\ 1}\underline{e}_{m-1}$

Therefore $\underline{e}_{m-1} = U_{m-1}^{-1}\underline{d}_{m-1}$

Note that y is real because $\det(U_m U_m^T) = \det(U_{m-1})^2\ y^2$

$$= \det(A_m)$$

and A_m is a positive definite matrix   In deriving the recursive form for U_m we have followed the method given in [74] (see page 423)

In terms of U we have to solve

$$U^T\underline{x} = U^{-1}H^T r \tag{1 5}$$

where U^T is a low r triangular matrix   The procedure of recovering data begins with the estimation of first data x_1 ich is given as

$$\tilde{x}_1 = z_1/u_{11}$$

where z_1 is the first element of $\underline{z} = U^{-1}H^T\underline{r}$ x_1 is obtained by passing $\tilde{x}_1$ through the nonlinear decision device   Then x_2 is obtained as

$$\tilde{x}_2 = (z_2\ u_{21}\hat{x}_1)/u_{22}$$

We continue this procedure till $\hat{x}_m$ is obtained   Figure 4 1 gives a block schematic of the data recovery scheme as d on the procedure just now outlined   In the figu e $\underline{u}_1$ denotes the ith row of U^{-1} and v_1 denotes ith row of $U^T - D$ wh re $D = \text{diag}(u_1$ $u_{mm})$ Although $\underline{u}_1$ and $\underline{v}_1$ are shown stored in separate register
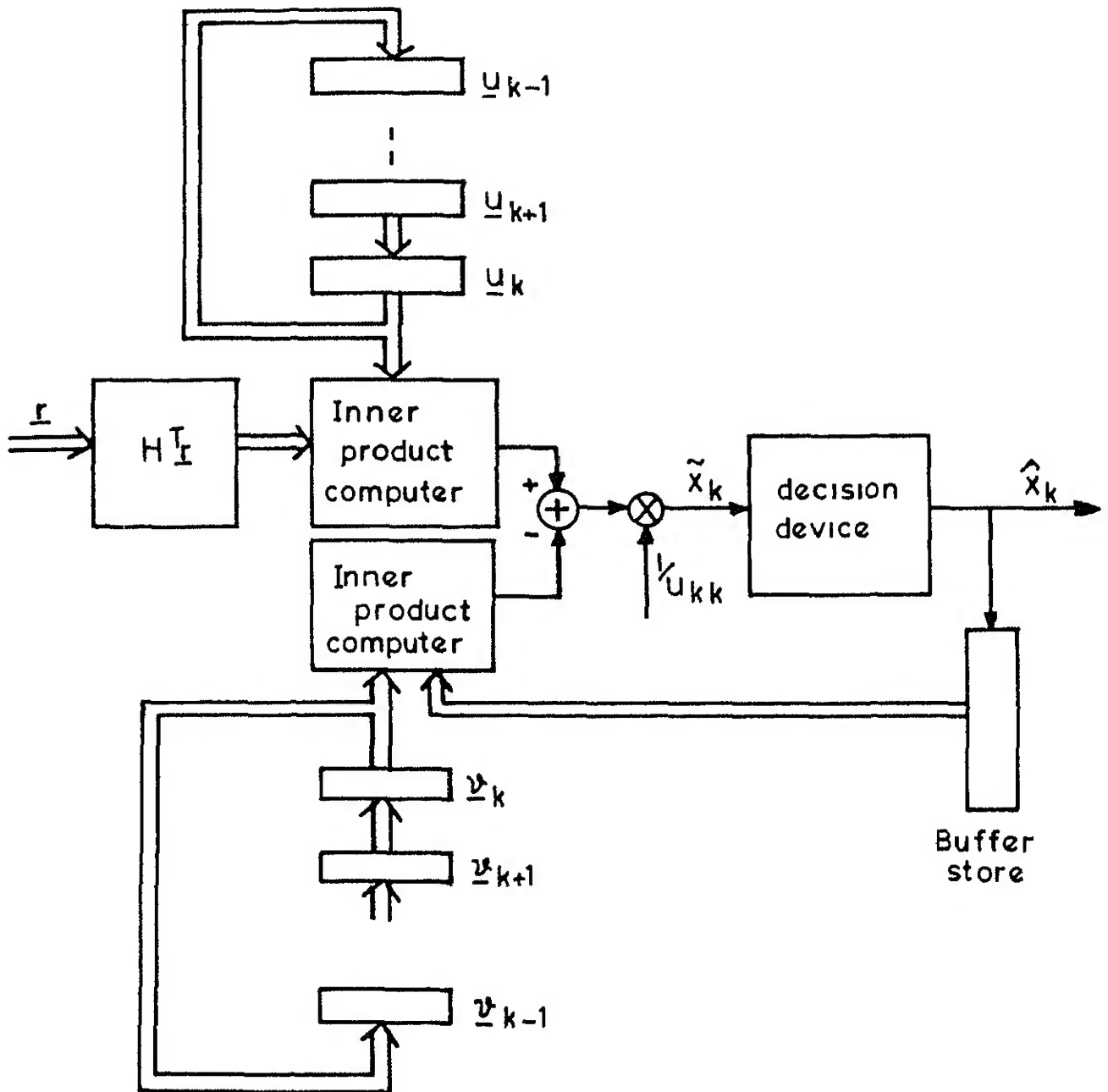
FIG 4 1 Block diagram of the data recovery scheme
for BDT using Cholesky decomposition

banks in the figure in all a set of m registers of length m are sufficient

Processing of each block consists of m cycl s In the figure we have shown the status of various registers for the kth data recovery For the next Jata (x_{k+1}) v_{k+1} takes the place of v_k and u_{k+1} take the place of $\underline{u}_k$ u_{kk} becomes $u_{k+1\ k+1}$ At the end of m cycles the in tial state is restored and the processor is ready to d al with the next block The inner product computer shown is the major computation block in the scheme In the case of binary data tl inner product computer used in the feedoack loop aoes not require multipli cations

Computational requirements of DF DP sch n for BDT using Chole ky decomposition method is same as that of DF-DR sche e for CSDT if the block leng h is equal to the number of forvard tap On th other hand the scheme shown in Fic 4 1 requires more storage compared to DF-JR schemes for CSDT syst ms How- ver with the present day t chnology storage requirement is not a crucial factor Whereas long bursts of error in the cnsc of CSDT due to error propagation especially in the medium S√R range is definitely a matter of concern In the case of DF-DP scheme for BDT systems error propagation is restricted to a block length in the worst situation

Analytical evaluation of the performance of DF-DR schemes is difficult due to the presence of nonlinear decision device However some idea of performanc can be obtained under the assumption of error-free feedback In this case probability of ith data symbol being in error $(PER)_i$ is given by

$$(PER)_i = \int_{(SNR)_i^{1/2}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2} y^2} dy = Q(\sqrt{(SNR)_i}) \qquad (4\ 6)$$

wher

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x e^{-\frac{1}{2} y^2} dy$$

$(SNR)_i$ can be computed as follows From (4 5) we see that the effective noise vector is given by

$$\underline{w} = U^{-1} H^T \underline{w}$$

Therefore SNR associated with ith data symbol is given by

$$(SNR)_i = u_{ii}^2 / (||\underline{u}_i||^2 N_0/2)$$

wher $||\underline{u}_i||^2$ denotes squared length of the ith row of $U^{-1}H^T$ and $N_0/2$ is the noise variance

4 2 MSE DECISION FEEDBACK DATA RECOVERY SCHEME

In the previous section we described one method of obtaining the parameters of DF-DR sch ne using Cholesky decompo ition in which we did not take noise variance into account We now derive a DR schem based on minimum mean square

error (MMSE) criterion   This scheme accounts for the noise
variance

   We first define several vec ors and matrices which will be
used in the following derivation

$\underline{b}$ $= H^T r$

$x_<$ $= (x_1\ x_2\qquad x_{k\ 1})^T$

v_k $= (v_{k\ 1}\ v_{k\ 2}\qquad v_{k\ k-1})^T$

u_k $= (u_{k\ k}\qquad u_{k\ m})^T$

H_k $= H$ with first $l\ 1$ olumns removed

$\underline{h}_k$ $= k$th column of H

$\tilde{H}_k$ $= H$ with first $k-1$ columns

   The objective is to obtain a set of coefficients $u_{k\ k}$
$u_{k\ k+1}\qquad u_{k\ m}\ v_{k1}\ v_{k2}\qquad v_{k\ k-1}$ such that estimate
of the kth data symbol given by

$$\hat{x}_k = u_{k\ k}\ b_k\ +\ +u_{k\ m}\ b_m\ -\ v_{k\ 1}\ x_1\qquad -v_{k\ k-1}\ x_{k-1}$$

$$(4\ 7)$$

is closest to th actual data x_k in tne mean square sense
That is

$$MSE_k = E[(\hat{x}_k - x_k)^2]\ \text{is minimum}$$

Here E[ ] denotes the standard exp ctation operator   Using
th principle of orthogonality in MSE minimization [7] we
obtain

$$F[e_k b_j] = 0 \quad j = x \quad x+1 \quad m$$

and $\quad L[e_\zeta x_1] = 0 \quad 1 - 1\,2 \quad k\,1$

where $e_k = x_k - x_k$ Using the fact that $b_j = h_j^T Hx + h_j^T \underline{w}$ we get

$$E[e_k b_j] = E[(\underline{u}_k^T H_k^T Hx + \underline{u}_k^T H_1^T \underline{w} - \underline{v}_k^T \underline{x}_k \quad x_k)(h_j^T H\underline{x} + \underline{h}_j^T w)]$$

$$= \underline{u}_k^T H_k^T H H^T h_j \quad \underline{v}_k^T \tilde{H}_k^T \underline{h}_j - h_1^T \underline{h}_j + \frac{N_o}{2} \underline{u}_k^T H_k^T \underline{h}_j = 0 \qquad (4\ 8)$$

$$j = k \quad k+1 \quad m$$

Similarly $\quad E[e_k x_1] = 0 \quad$ gives

$$\underline{u}_k^T H_k^T \underline{h}_1 \quad v_{k\,1} = 0 \qquad 1 = 1\,2 \quad k-1$$

$$v_{k1} = \sum_{j=k}^{m} u_{kj}\, a_{j-1} \quad \text{where} \quad a_{j-1} = h_j^T h_1$$

In th matrix vector notation

$$v_k = \tilde{H}_k^T H_k u_k \qquad\qquad (4\ 9)$$

Substituting (4 9) in (4 8) and rearranging terms we obtain

$$\underline{u}_k = [H_k^T[HH^T + \frac{N_o}{2} I - \tilde{H}_k \tilde{H}_k^T]H_k]^{-1} H_k^T \underline{h}_k$$

$$= [H_k^T H_k + \frac{N_o}{2} I]^{-1} (H_k^T H_k)^{-1} H_k^T \underline{h}_k \qquad\qquad (4\ 10)$$

But $(H_k^T{}_k)^{-1} H_k^T h_k = (1 \cup 0 \qquad 0)^T$

Therefore $\underline{u}_k$ is obtained as the first column of

$$[H_k^T H_k + \frac{N_o}{2} I]^{-1}$$

Finally the resulting MSE associated with the kth data is given by

$$MSE_1 = -E[e_k x_k] = 1 - \underline{u}_k^T H_k^T h_k = 1 - \sum_{i=k}^{m} u_i \ i \ i \ k$$

$$= 1 - \sum_{j=0}^{g} a_j u_{k \ j+k} \qquad (4 \ 11)$$

The mean square error given by (4 11) reflects to a great extent the quality of ne DR scheme It an be considered as a mea ure of the performance for it indicates the variance of the error If MSE is significant probability of the kth data symbol being in error is large Thus $1/(MSE)_k$ may be considered as a measure of effective SIR associa ed with the kth data symbol

Components of u_k can be computed recursively u ing Trench algorithm [75] because the matrix $[H_k^T H_k + \frac{N_o}{2} I]$ is a positive definit Toeplitz matrix First we cal ulate u_m given by

$$u_n = u_{mm} = (H_m^T H_m + \frac{N_o}{2} I)^{-1}$$

where $H_m^T H_m = \underline{h}_m^T h_m = h_o^2 + h_1^2 + \qquad h_g^2$

The oth r $\underline{u}_k$ s k = n-1 m-2 1 can be recursively obtained by following the procedure given in [75] Thus the computational complexity is $O(m^2)$ as the procedure for

calculating $\underline{b}_i$ $i = 1\ 2$ m is same as inverting a mxm

positive def nite Treplitz matrix by Trench algorithm   The

implementation scheme is same as the one shown in Fig 4 1

except that $\underline{u}_i$ and $\underline{v}_i$ are given by (4 9) and (4 10) res-

pectively

4 3 DECISION FELDBACK SCHL E BASED ON MINIMU PPOPERTY
OF FOURIEI EXPANSIONS

So far in our analysis no thought has been given to the

problem of illconditioning o1 the channel autocorrelation

matrix $H^T H$ $H^T H$ 1 said to be illcond tioned if the ratio

of maximum ard minimum eigenvalue is large [74]   When $H^T H$

is illconditioned the solution obtained oy inverting $H^T H$ is

v 1y sensitive to truncation errors and noise in the received

vector   It is therefore essential to consider numerically

stable methocs   Towards this end in this section a numeri-

cally stable method of solving linear system of equat ons

ba ed on minimum property of Fourier expansion due to

Jang [58] is dopted to derive a decision feedback data

recovery cheme

Minimum Property of Fourier Expansion   For an/ finite or

infinite lincarly independen sequence of el ments $\alpha_1\ \alpha_2$

of a Hilbert space S_H there always exists an orthonormal

system β_1 β_2 such that

$$\beta_{j}^{t} = \beta_j / ||\beta_j|| \quad j = 1\ 2$$

where $\beta_1 = \alpha_1$ $\beta_{j+1} = \alpha_{j+1} - \sum_{k=1}^{j} (\sigma_{j+1} \ \beta_k)\beta_k / ||\nu_k||^2$ (4 12)

$$j = 1 \ 2$$

This procedure is known as Gram-Schmidt orthonormalization
Here $(\alpha \ \beta)$ denote inner product of α and β

Consider an element α in S_F Then x_j s which minimize the norm

$$||\alpha \ - \ \sum_{j=1}^{n} x_j \beta_j^*||$$

are known as the Fourier coefficien s of α They are given by

$$x_j = (\alpha \ \beta_j^t)$$

This property of x_j s is known as the minimum property of Fourier expansion

Now let $\underline{z}_1 \ z_2$ $\underline{z}_m$ be the m column vectors of H in (4 1) Since the column of H are linearly independent orthonormalizing $\underline{z}_1 \ \underline{z}_2$ $\underline{z}_m$ we obtain $\underline{y}_1^* \ \underline{y}_2^*$ $\underline{y}_m^*$ Then we have the following result due to Wang [58]

Theorem 4 1 Let r given in (4) be an el ment of an n-dimensional Hilb rt space $\tilde{x}_j$ s which minimize the norm $||r \ \sum_{j=1}^{m} x_j \underline{z}_j||$ are given by

$$\tilde{x}_m = (r \ \underline{y}_m)/||y_m||^2$$

$$\tilde{x}_{m \ j} = (\underline{r} - \sum_{k=m-j+1}^{n} \tilde{x}_k z_k \ y_{m-j})/||y_{n-j}||^2 \qquad (4 \ 13)$$

$$j = 1 \ 2 \qquad m-1$$

(Proof of this theorem is not given in [58] Therefore for the sake of completeness we prove the theorem in the following manner)

Proof   We know that given r and y_j s the c_j s which minimize the norm

$$|| r - \sum_{j=1}^{m} c_j y_j ||$$ are given by

$$c_j = (r \ y_j)/||y_j||^2$$

By substituting for y_j in terms of z_j s using (4 12) and grouping the terms common to z_j s we can obtain the recursive equations given by (4 13)   To see this consider

$$\sum_{j=1}^{m} c_j y_j$$ and substitute for y_j using (4 12)   This gives

$$\sum_{j=1}^{m} c_j y_j = c_1 z_1 + c_2(z_2 - \frac{(z_1 \ y_1)z_1}{||y_1||^2}) + c_3(\ - \frac{(z_3 \ y_1)z_1}{||y_1||^2}$$

$$- \frac{(z_3 \ y_2)y_2}{||y_2||^2}) +$$

$$+ c_m(z_m - \frac{(z_m \ y_1)z_1}{||y_1||^2} \ - \frac{(z_m \ y_{m-1})y_{m-1}}{||y_{m-1}||^2})$$

From the above expression it is clear that the coefficient of z_m is c_m Thus

$$\tilde{x}_m = c_m = (r \ y_m)/||y||^2$$

The coefficient of z_{m-1} gives

$$x_{m-1} = (r - x_m z_m \quad y_{m-1})/||y_{m-1}||^2$$

The rest of the coefficien s can be similarly obt ined

It may be noted that in th method discus ed above x_m is obtained first which is used in obtaining x_{m-1} by back substitution However for the sake of uniformity we wish to obtain x_1 first This can be easily don by orthonormalizing the columns of H starting from the mth column Le y_1 y_2 y_r denote the orthogonal columns ob ain d this way Then the governing equations of the data recovery s heme are given by

$$\tilde{x}_1 = (r \quad y_1)/||y_1||^2$$

$$\tilde{x}_1 = (r - \sum_{k=1}^{1-1} \hat{x}_k h_k \quad y_k)/||y_1||^2 \qquad (4\ 14)$$

$$i = 2\ 3 \qquad m$$

where h_k is the kth column of H $\hat{y}_1$ $1 = 1\ 2$ m are the es mates of the data values obtained by pa sing $\tilde{x}_1$ $1 = 1$ m through a nonlinear decision device

Figure 4 2 shows he data recovery sch me Processing of each block of data consists of m cycles lajor computational load is due to the inner product computation in every cycle
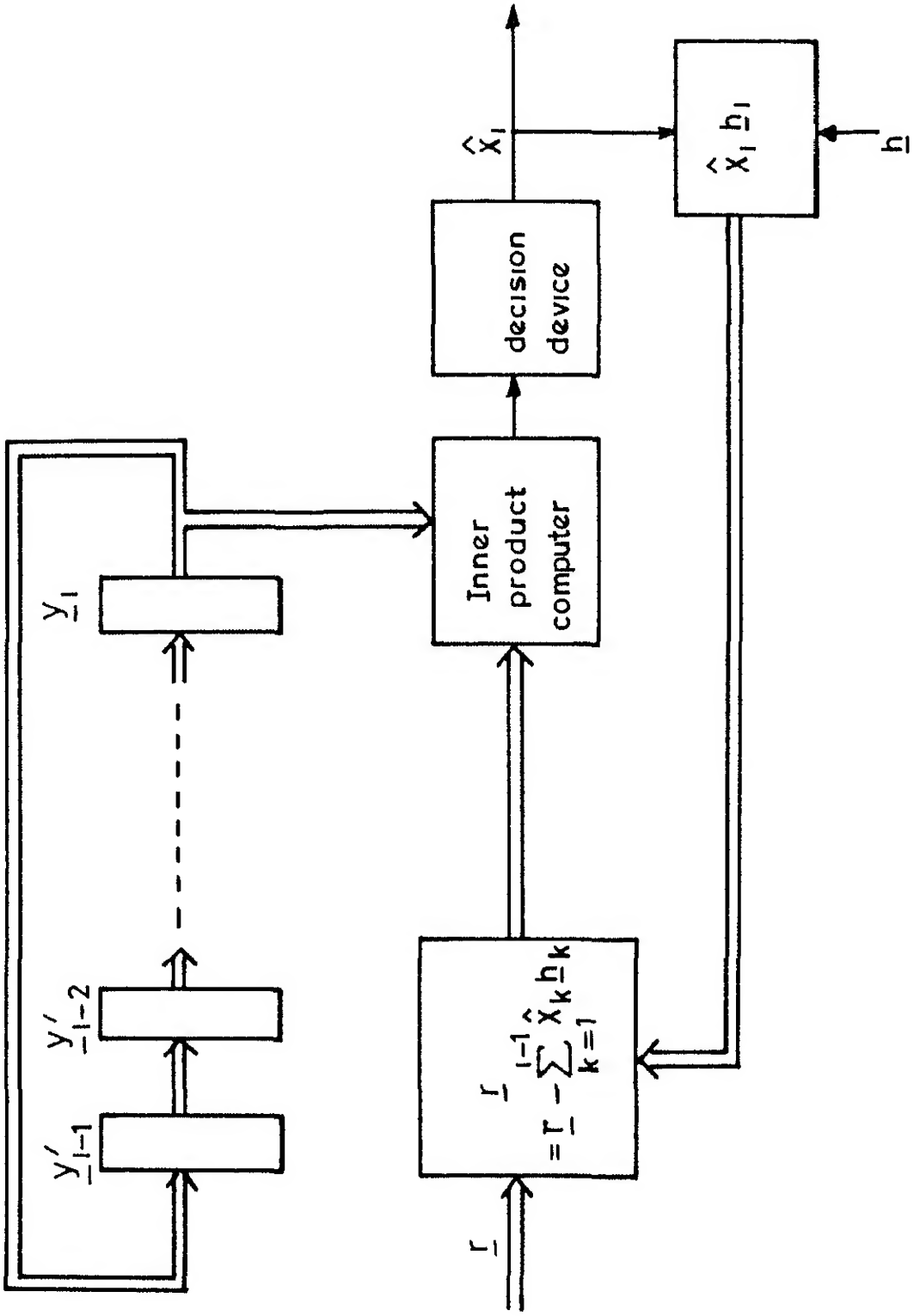
FIG 4 2 Block diagram of the data recovery scheme using minimum property of Fourier expansion

The performance of the sch me can be analytically evaluated in th absence of error   The probabilit/ of error associated with ith da a symbol is given by (4 6)   The ωNR associat d ω th the ith data symbol can be calculated as follows   Substituting for r from (4 1) in ($\sim$ 14) for x_1 we get+

$$\tilde{x}_1 = (Hx + w \quad _{\angle_1})/||y_1||^2$$

$$= (\sum_{j=1}^{m} x_j (h_j \ y_1) + (w \ y_1))/||y_1||^2$$

Substituting for h_j in t rms of y_j using (4 15) we see that

$$\tilde{x}_1 = x_1 + (y_1 \quad w)/||y_1||^2$$

Similar expressions are obtained in the ca e of other data symbols al o   Thus

$$(SNR)_1 = ||y_1||^2/(N_o/2)$$

$||y_1||^2$ $i = 1 2$ m thus denotes a m asure of the performance of the scheme and it is related to the c annel sample r sponse vector   When the columns of the channel convolutional matrix H are orthogonal $||y_1||^2 = 1$ for $i = 1 2$ m if $h^T h = 1$ When corr lations exist among he columns $||y_1||^2 \leq 1$ $i = 1 2$ m   This can be seen by noting that

$$y_m = h_m$$

and $\underline{y}_{m-1} = \underline{h}_{m-1} - \sum\limits_{k=m-1+1}^{m} (\underline{h}_{m\,1}\,\underline{y}_l)\underline{y}_k/||\underline{y}_k||^2 \quad 1 = 1\,2 \qquad m-1$

$$(4\ 15)$$

From $(4\ 15)$ we obtain $||\underline{y}_m||^2 = 1$ and

$$||\underline{y}_{m\,1}||^2 = 1 - \sum\limits_{k=m-1+1}^{m} (\underline{h}_{m-1}\quad\underline{y}_k)^2/||y_k||^2 < 1$$

4 4 DECISION FEEDBACK SCHEME BASED ON MODIFIED AUSTIN S APPROACH

On of the most widely used DR scheme in the context of CSDT sy tem is the decision feedback equali er Austin [13] has deriv d the equations of decision feedback equalizer for C DT systems by making suitable assumptions so that detection theory results can be easily applied As a counterpart of this scheme in this section w derive a DF-DR scheme by modifying Austin s assumptions In the following discussion we assume binary data transmission

Let x_k denote the symbol on which decision is to be made We modify Austin s assumption in the following manner for the case of BDT

1   Data symbols are independent Gaus ian variables with zero-mean and unit variance for $i > k$

2   x_1 are known for $i < k$

3   Noi e components ar independent Gaussian variables with zero mean and variance $N_o/2$

4   x_k takes 1 or -1 value with equal piobability

Th 1 kelihood ratio for x_k is given by

$$L_k = \frac{p[\underline{r}/\underline{x}_k^p \; x_k=1]}{p[\underline{r}/x_k^p \; x_k=-1]}$$

wher $\underline{x}_k^p = (x_1 \; x_2 \quad\quad x_{k-1})^T$ Let $x_k^f = (x_{k+1} \; x_{k+2} \quad x_m)^T$ denote the future data value with respect to x_k $p(\;)$ denotes probabili y den it/ function Now L_t can be alternatively expressed as

$$L_k = \frac{\int p(\underline{r}/x_k^p \; x_k^f \; x_k) \; p(\underline{x}_k^f) \; dx_k^f \; |_{x_k=1}}{\int p(r/x_k^p \; \underline{x}_1^f \; x_k) \; p(x_k^f) \; d\underline{x}_k^f \; |_{x_k=-1}}$$

Making use of the a sumptions made in the beginning of this s ction ve can write

$$p(\underline{r}/\underline{x}_k^p \; \underline{x}_k^f \; x_k) = k_1 \; \exp[\sum_{i=1}^{1} x_i b_i \quad \sum_{i=1}^{n} \sum_{j=1}^{n} \gamma_i x_j \; a_{i-j}] \qquad (4 \; 16)$$

where K_1 is a constant and

$$b_i = \frac{2}{N_o} \sum_{j=0}^{g} h_j \; r_{j+i} \qquad i = 1 \; 2 \qquad m \qquad (4 \; 17)$$

$$a_i = \frac{2}{N_o} \sum_{j=0}^{g} h_j \; h_{j+i} \qquad i = 0 \; 1 \qquad m-1 \qquad (4 \; 18)$$

Dumping all the factors which cancel out while forming L_k into K_2 we re rite (4 16) as

$$p(r/\underset{\sim}{x}_k^p \; x_r^f \; x_k) = K_2 \; \exp[x_1 b + \sum_{1=k+1}^{m} x_1 b_1 \quad 2 \sum_{1=1}^{k-1} \sum_{j=k+1}^{m} x_1 x_j a_{i-j}$$

$$- 2x_k \sum_{1=1}^{k-1} x_1 a_{1-k} - 2x_k \sum_{j=k+1}^{m} x_j a_{k-j}$$

$$\sum_{1=k+1}^{m} \sum_{j=k+1}^{m} x_1 \; j \, a_{1-j}]$$

Using assumption (1) we also have

$$p(x_k^f) = K_3 \; \exp[\; \frac{1}{2} \sum_{1-k+1}^{m} \sum_{j=k-1}^{m} x_1 x_j \; \delta_{1j}]$$

where $\delta_{1j} = \begin{cases} 1 & \text{if } 1 = j \\ 0 & \text{if } 1 \neq j \end{cases}$

U ing the structure of multidimensional Gaussian density function and following Austin we can obtain the numerator of L_k as

$$\exp[x_k b_k - 2x_k \sum_{1=1}^{k-1} {}_1 a \quad k + \sum_{j=k+1}^{m} \sum_{1=k+1}^{m} (\frac{1}{2} b_j - \sum_{n-1}^{k \; 1} x_n a_{j-n}$$

$$- x_k a_{j-k}) \; q_{j1}^k (\frac{1}{2} b_1 - \sum_{n=1}^{k-1} x_n 1_{1-n} - x_k a_{1-k})]$$

where q_{j1}^k are the elements of m tri $Q_k = R_r^{-1}$ The elements of R_k ar given by

$$R_{1j}^k = a_{1-j} + \frac{1}{2} \; \delta_{ij} \qquad 1 \; j > k \tag{4 19}$$

Thus we obtain

$$L_k = \exp[2b_k - 4 \sum_{i=1}^{k-1} x_1 a_{1-k} \quad 4 \sum_{n=k+1}^{m} a_{n-k} [\sum_{j=k+1}^{n} (\frac{1}{2} o_j -$$

$$\sum_{1=1}^{k-1} a_{1-j} x_1) q_{j,n}^k]$$

Now let $g_{kj} = - \sum_{1=k+1}^{m} q_{1j}^k a_{1 k}$ $n = 1\ 2$ $m\ 1$ (4 20)

$$j = k+1 \quad m$$

and $f_{kj} = 2a_{j k} + 2 \sum_{1=k+1}^{m} g_{k1} a_{1-j}$ $k = 2\ 3$ m (4 21)

$$j = 1\ 2 \quad k-1$$

The decision rule in terms of g_{kj} s ard f_{kj} s become

$$b_k \quad \sum_{j=k+1}^{m} g_{kj} b_j - \sum_{=1}^{k-1} f_{k1} x_1 \quad \mathop{\gtrless}_{x_k=-1}^{x_k=1} \quad 0 \tag{4 22}$$

$$k = 1\ 2 \quad m$$

Th decision rule given by (4 22) clearly reflects the structure of the data recovery schen The structure is shown in Fig 4 3 In the figure

$$b = (b_1\ b_2 \quad b_m)^T \quad g_k = (g_{k\ k+1} \quad g_{k\ n})^T$$

$$f_k = (f_{k1}\ f_{k2} \quad f_{k\ m-1}) \quad b_k = (b_{k+1}\ b_{k+2} \quad b_m)^f$$
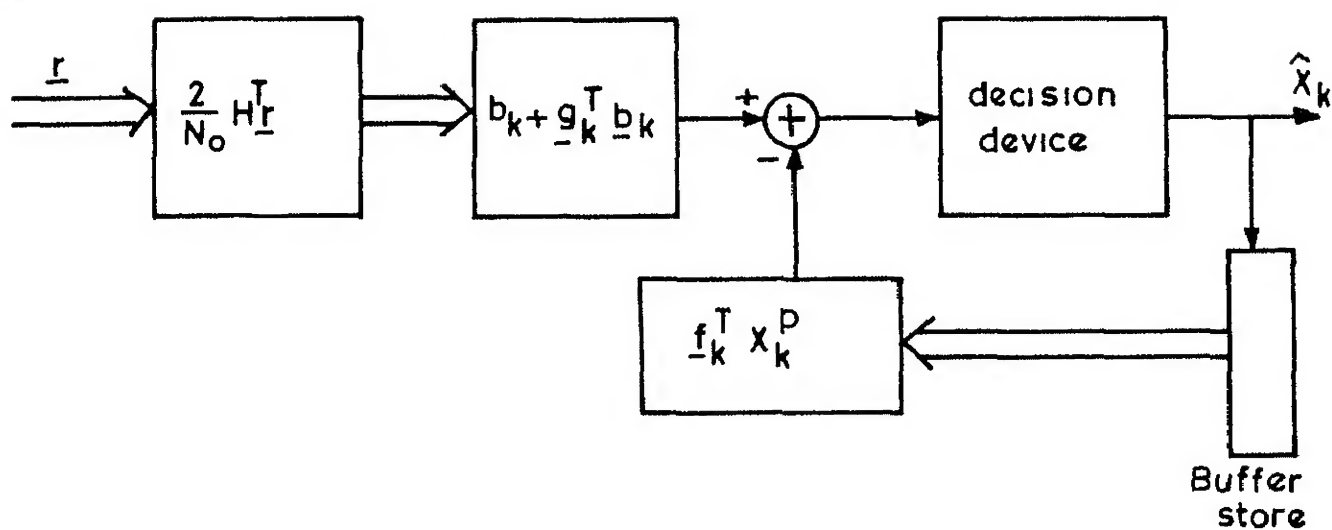
FIG 4 3   Block diagram of the data recovery scheme
using modified Austins approach

In order to make decision regarding the kth data symbol we need the knowledge of g_{ij} $j = k+1$ n and f_{kj} $j = 1 2$ $k-1$

We illustrate the for going discussion with an example

Example 4 1 Consider a block containing three data values $x = (x_1\ x_2\ x_3)^1$ transmitted over a channel having $(h_0\ h_1\ h_2)$ a its finite impulse response sequence Then or a received vector $r = (r_1\ r_2\quad r_5)^T$ we have

$$a_0 = \frac{1}{_0} (h_0^2 + h_1^2 + h_2^2) \qquad b_1 = \frac{2}{N_0} (r_1 h_0 + r_2 h_1 + r_3 h_2)$$

$$a_1 = \frac{1}{_0} (h_0 h_1 + h_1 h_2) \qquad b_2 = \frac{2}{N_0} (r_2 h_0 + r_3 h_1 \quad r_4 h_2)$$

$$a_2 = \frac{1}{N_0} (h_0 h_2) \qquad\qquad b_3 = \frac{2}{_0} (r_3 h_0 + r_4 h_1 + r_5 h_2)$$

The matrices R_k $k = 1 2$ are given b

$$R_1 = \begin{vmatrix} a_0 + \frac{1}{2} & a_1 \\ \\ a_1 & a_0 + \frac{1}{2} \end{vmatrix} \qquad R_2 = [a_0 \quad \frac{1}{2}]$$

Let

$$\Psi_1 = \begin{bmatrix} q_{22} & q_{23} \\ \\ q_{32} & q_{33} \end{bmatrix} \qquad \text{and} \qquad \Psi_2 = q_{33}$$

Then

$$g_{23} = -q_{33}a_1 \qquad\qquad f_{21} = 2a_1 + 2g_{23}a_2$$

$$g_{12} = -(q_{22}a_1 + q_{23}a_2) \qquad\qquad f_{31} = 2a_2$$

$$g_{13} = -(q_{32}a_1 + q_{33}a_2) \qquad\qquad f_{32} = 2a_1$$

The decision rul (4 22) becomes

$$b_1 + g_{12}b_2 \quad g_{13}b_3 \quad \overset{x_1 = 1}{\underset{x_1 = -1}{\gtrless}} \quad 0$$

$$b_2 + g_{23}b_3 - f_{21}x_1 \quad \overset{x_2 = 1}{\underset{x_2 = -1}{\gtrless}} \quad 0$$

$$b_3 \quad f_{31}x_1 - f_{32}x_2 \quad \overset{x_3 = 1}{\underset{x_3 = 1}{\gtrless}} \quad 0$$

4 1 Co putation of the Tap Coe ficients

Ihe co fficients g_{KJ} $J = k+1$ m c n be viewed as the
 co f icients of the forward fil er in he cas f x_k More-
ov r these co fficient can be calculated efficiently using
Levinson-Durbin algorithm [7]   ior that pu pose (4 20) is
r written below in the matrix vector form

$$\underline{g}_k = - \langle_k \underline{a}_k$$

here $\underline{g}_k = (\sigma_{k\ k+1} \qquad g_{k\ 1})$ and $\underline{a}_l = (a_1\ a_2 \qquad a_{m-k})$ Since

Q_k is the inverse of P_k we can write $\underline{g}_k$ as the solution of

$$P_k \underline{g}_k = -a_k$$

where R_k is defined by (4 19)

R_k is a symmetric Toeplitz matrix Therefo   ve can make use of Levinson–Durbii recur ive algorithm given in [7] for obtaining $\underline{g}_k$ s   The required st ps for computing g_k s are given below

1   a_{11} $- a_1/(_o + \frac{1}{2})$ $_o = a_o + \frac{1}{2}$

2   d_{11} $= (a_1 - \sum\limits_{j=1}^{1-1} d_{1-1\ j}\ a_{1-j}) /e_{1-1}$

3   d_{1j} $= d_{1-1j} - u_{11}\ d_{1-1\ 1\ j}$ $1 < j < 1-1$

                                                $1 = 2\ 3$ $k-1$

4   e_1 $= (1\quad d_{11}^2)\ _{1-1}$

5    $g_{n-1\ m}$ $= -d_{11}$

6    $g_{k\ k+1}$ $= d_{n-1\ 1}$ $1 = 1\ 2$ $m\ {}^{\backprime}$ $k = 1\ 2$ $m-1$

4 4 2 Modifiea Au tin s Approach and Nonlinear Gaussian
       limina ion I thod

In this sect on we sho/ that the modified Austin s approach may be vi wed as nonlinear Gaussian elimination method vith approoriate odificitions For hi purpose (4 22) may be written using ma rix vector no ation a

$$\begin{bmatrix} 1 & g_{12} & g_{13} & & g_{1n} \\ 0 & 1 & g_{23} & & g_{2m} \\ 0 & 0 & 1 & & g_{3m} \\ & & & & \\ 0 & & & & 1 \end{bmatrix} \begin{bmatrix} b \\ b_2 \\ b_3 \\ \\ b_n \end{bmatrix} \begin{bmatrix} 0 & 0 & & 0 \\ f_{21} & 0 & & 0 \\ f_{31} & f_{32} & & 0 \\ & & & \\ f_{n1} & f_{m2} & f_{m\,m\,1} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ y_2 \\ x_3 \\ \\ x_m \end{bmatrix} \gtrless \underline{0}$$

$$Gb - \Gamma x \gtrless 0 \qquad\qquad (4\ 23)$$

where $\underline{0}$ is a null vector and $\gtrless$ denotes symbolically the non-linear d cision de ic Note that th right hand si e of (4 22) is an esti ate of x_k Therefore th right hand side of (4 23) i an est i at of t data vector x That is

$$Gb - Fx = I_m \underline{x}$$

L t $\Gamma = F + I_m$ Th n the above equation can be wri ten as

$$G^{-1} \Gamma x = b$$

(Γ is invertible since the diagonal elements ar nonzero)

ut $\underline{b} = H^T \underline{r}/(N_0/2)$ which gives

$$\frac{2}{N_0} G^{-1} \Gamma \underline{x} = H^T r \qquad\qquad (4\ 24)$$

Comparin (4 24) wit (4 2) it can be seen that $\frac{2}{N_0} G^{-1} \Gamma$ plays the role of $H^T H$ Thus w may call $\frac{2}{N_0} G^{-1} F$ as eff c ive autocorrela-tion matrix With this int rpr tation see that modified Aus in s approach is equivalent to nonlinear Gauss an elimination method

4 5 SOME OBSERV TIONS

As we are dealing with the problem of solving lin ar equa-
ors in the presence of noise when the unknown variables take
v us from a finite set it is natural to think tha available
methods for solving linear equations must apply with approori te
modifications   We demonstrated this b considering four DF-DR
schemes which are centrally oased on tl   dea of nonlin ar
Gaussian elimination method

The two DF-DR chcmes considered in Section 4 1 nd
S ction 4 3 essentially modify already existing ne lo   for
solving linear equations by incorporatin, l cision feedback as
explained in he beg ing of the chapter However the methods
discussed in o ctions 4 2 nl 4 4 re derived based on a set of
as umpt ons and a performance criterion   DF DR sche ne of
S ction 4 2 emoloys mathem tically tractabl   MMSE criterion
while in Section 4 4 we used nore difficult e   on eory
approach   In this approach he complexity of the scheme is
lleviated considerably by assuming that the data value are
indep ndent G ussian variables The major nderlying assumption
in both the sch m s is that the past decisio s are rro ree

In the absence of noisc all the four sch es no aoupt
provice error free perfor ance even withou th nonlinear
decision device how far d c sion f edbacl nelps to improve

the performance ir the pres nce of noise can only be evaluated with the help of simulation Although decision edback is an old idea [77] no useful mathematic l analysis is ava_lable which deals with analytical evaluation of performance The majo tumbling block appears o be th nonlin ar decision evice which d fies y analysis In Chapt r 8 we give simula ion results on DF DF scho es a veloped in sectiors 4 3 ard 4 4

CHAPTL 5

DATA R COVERY SC IEFL USING k CI CULANT COMPLETIONS

In he previous chapter we made use of past decisions and band d Foeplitz natuie of the ciannel autocorrelation matrix H^TH for deriving decision f edback deconvolution schemes which are suboptimal compared to t iL dc onvolution scheme of Chapter 3 It may be o s rved ha so far in our study no attention has been given to the convolution matrix H of the channel Ve shall see in this chap er that H has some int rest- ng propertie compired to H^Tr from the point of view of ~educing computations and storag requir ments Ba ed on k- irculant conoletion of P we develop two data iecovery scrcnes Thesc schemes are easily implementablc and have reduced comouta tional complexity Turthermore unlike the DF DP schemes of Chapcer 4 which are nonlinear the DR schenos studied in this chapter are linear an they permit analytical evaluation of ior performance

The chapt r is organiz d as follows In Section o 1 we i roduce the notion of k-C rculant co oletion and describe its application to deconvoluti n Tio nethods of inverting l- Circulant matric s are gi er in Section o 2 The First method us s FFT for inv rting k-C i culant com letion of H Complexity

of his algorithm is $O(n\log_2 n)$ wh re n is the number of rows of H Th second method makes use of o rsit/ nd block k-Circulant nature of the r-Circulan compl tion of H to pro- vid an $O(n\log_2 g_1)$ algorithm where $g_1 = g\ 1$ In S ction 5 3 we discu s two DF schc n s which are based on k-DFT domain d convolution discussed in Sec ion 5 1 and direct inversion of c Circulant completion of H d scribed in c ion b 2 In Section 5 4 w obtain cyplicit e pression for probability o rror and compare it with the performance r DF scheme based on least quare d convolution netl od

5 1 F-CI RCULANT COMPLLTIO.J AI D FI1 1TE DECON VOLUTION

Consider the received vector $\underline{r}$ given by (2 7)

$$\underline{r} = Hx + w \tag{5 1}$$

When no statistical charactc ization of $\underline{x}$ and w are available we know that the maximum likelihood estimat of x under the assumption of r al valued daca is giv n by

$$x_L = (H^T H)^{-1} H^T \underline{r}$$

The most time con uming oo ration in ob t 111g x_{LS} is the inv r ion of $H^T H$ Ev n ir we assume that $(H^T H)^{-1} H^T$ is computed before the transmission of data memory requirements can be come prohibitive Therefore even with the availabili y of s veral fast algorithms for inverting $H^T H$ [78] [80] there is

still need or processing algori has th⁺ are easily implemen

table   In this section /e consider the applic tion of k

Circulant matrices [81] [82] for olving deconvolu on pro-

bl m   ior he sake of e sy reference so ie of the results

r garding k-circulant na rices are given in ppendix D

As an illustration we consider $(\jmath\ 1)$ and it 1-Circulant

count-rpart for $n = 5$ $m = \jmath$ and $g = 2$ In this case $(\jmath\ 1)$

can be written as

$$
\begin{bmatrix} r_0 \\ r_1 \\ r_2 \\ r_3 \\ r_4 \end{bmatrix} = \begin{bmatrix} h_0 & 0 & 0 \\ h_1 & h_0 & 0 \\ h_2 & h_1 & h_0 \\ 0 & h_2 & h_1 \\ 0 & 0 & h_2 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} h_0 & 0 & 0 & kh_2 & kh_1 \\ h_1 & h_0 & 0 & C & kh_2 \\ h_2 & h_1 & h_0 & 0 & 0 \\ C & h_2 & h & h_0 & 0 \\ 0 & U & h_2 & h_1 & h_0 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} w_0 \\ w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix}
$$

$$ \underline{r} = H_1 \underline{x} + w \tag{5 2} $$

where H_k 1 the k-Circulant completion of H and x_e is obtained

by padding x with two zeros It 1 not d) icult to see that

the r oresentation giv n in (5 2) in terms of H_k is also valid

in the general case when r is fini e $(n = m+g)$ and x is padded

with n-m zeros to obtain x_e

From row on we call H_1 the Circulant completion of H as

it is easy to sc~ from (5 2) that H_1 is a Circulant matrix

Similarly H_{-1} will denote the Skew Circulant completion of H

5 1 1   Deconvolution Usin   Circulant Completion

Consider the circulant completion of $\llcorner$ If P $\underline{X}$ $\neg a$ $\underline{y}$ deno e DFT of $\underline{r}$ x and w respectiv ly using tne prope ty of cyclic convolution [59] we can write (5 2) as

$$\underline{R} = H \, \diagup + \underline{W} \qquad\qquad (5\ 3)$$

where $\underline{H}$ denote the DFT of the first column of H (H $\underline{X}$) denotes poin wise multiplication of tl $\cap$ compon nts of the individual vectors   Equation (5 $\cup$) suggests an easy way of decon olving whcnever none of the component of $\underline{H}$ is too small or zero   In su h cas $\cap$s an estimate of x can be obtained as

$$\underline{x}_{1e} = F^{-1}(\underline{R}/H) \qquad\qquad (5\ 4)$$

where F denotes the DFT matrix of order n and $(\underline{R}/\underline{H})$ is a vector obtained by componentwis division   This method or deconvolution can be attributed to Cooley [83] who after becoming instrumental in th development of FFT algorithm [84] proposed the foregoing approach as an application of FFT in the context of signal process ng   Some of the problems associated with such straightforward application to situat ons involving erroneous measure nents hav been discu sed t length by Hunt[59]

5 1 2   Deconvolution Using k-Circulant Completion

In this section we show that (5 3) and (5 4) can also be generalized to the cas of k-Circular completions   For this

purpose let $\underline{b} = (b_0 \; b_1 \qquad \jmath_{n-1})^T$ and $c = (c_1 \; c_1 \qquad c_{n\;1})^T$

be two vec ors   Ie define k-cyc ic convolution of $\jmath$ and $\underline{c}$ as

anothe vector d given by

$$d_1 = \sum_{\jmath=0}^{1} b_\jmath \;\; {}_{1-\jmath} + k \sum_{\jmath=1+1}^{n-1} b_\jmath \; c_{(1-\jmath)\bmod n} \qquad (5\;5)$$

Ie define k-DFT of $\underline{b}$ an$\underline{\jmath}$ c as (see Appendix B)

$$B_{k1} = \sum_{\jmath=o}^{n-1} b_\jmath (\lambda w^1)^\jmath \qquad 1 = 0\;1 \qquad n-1 \qquad (5\;6)$$

$$C_{k1} = \sum_{\jmath-o}^{n-1} c_\jmath (\lambda w^1)^\jmath \qquad 1 = 0\;1 \qquad n-1 \qquad (5\;7)$$

respectively   The corresponding inverse tran forms are

$$b_1 = \frac{1}{r} \sum_{\jmath=o}^{n-1} B_{k\jmath} (\lambda w^\jmath)^1 \qquad 1 = 0\;1 \qquad n-1 \qquad (5\;8)$$

$$c_1 = \frac{1}{n} \sum_{\jmath=o}^{n-1} C_{k\jmath} (\lambda w^\jmath)^1 \qquad 1 = 0\;1 \qquad n-1 \qquad (\jmath\;9)$$

wi re λ is an nth root of 1 nd $w = e^{\jmath 2\pi/\jmath}$ Now we prove the
iollowing theo em

Theorem 5 1   If B_1 and C_k represent k-DFT of b and $\underline{c}$ resp cti
/ ly giv n by (5 6) and (5 7)

$$d_1 = \frac{1}{n} \sum_{\jmath=o}^{n-1} B_{k\jmath} \; C_{1\;\jmath} \; (\lambda w^\jmath)^{-1} \qquad 1 - 0\;1 \qquad n-1 \qquad (5\;10)$$

or $\qquad \underline{D}_k = (B_k \; C_k) \qquad\qquad\qquad\qquad\qquad (5\;11)$

where D_k is the k-DFT of d

Proo   Substituting for b_j and c_j from (5 8) and (5 9) in (5 5) we obtain

$$d_1 = \sum_{j=0}^{1} \frac{1}{n} \sum_{t=0}^{n-1} B_{kt}(\lambda w^t)^{-j} \frac{1}{n} \sum_{s=0}^{n-1} C_{ks}(\lambda w^s)^{(1-j)}$$

$$k \sum_{j=1}^{n-1} \frac{1}{n} \sum_{t=0}^{n-1} B_{kt}(\lambda w^t)^{-j} \frac{1}{n} \sum_{s=0}^{n-1} C_{ks}(\lambda w^s)^{-(1-j+n)}$$

Sinc $\lambda^n = k$ on r arrunging the above expres1on becomes

$$d_1 = \frac{1}{n^2} \sum_{j=0}^{1} \sum_{t=0}^{n-1} \sum_{s=0}^{n-1} B_{1t} C_{ks} \lambda^{-1} w^{-s} w^{j(s-t)}$$

$$+ \frac{1}{n^2} \sum_{j=1+1}^{n-1} \sum_{t=0}^{n-1} \sum_{s=0}^{n-1} B_{kt} C_{ks} \lambda^{-1} w^{s1} w^{j(s-t)}$$

Ch nging the order of summation it becomes

$$d_1 = \frac{1}{n^2} \sum_{=0}^{n-1} \sum_{s=0}^{n-1} B_{kt} C_{ks} \sum_{j=0}^{n-1} \lambda^{-1} w^{-s1} w^{j(s-t)}$$

But $\sum_{j=0}^{n-1} w^{j(s-t)} = \begin{cases} n & \text{if} \quad s=t \\ o & \text{if} \quad s \neq t \end{cases}$

Th refore $d_1 = \frac{1}{n} \sum_{s=0}^{n-1} B_{ks} C_{ks}(\lambda w^s)^{-1}$ $1 = 0\ 1 \qquad n-1$

$*$

Using (5 11) we easily ee that if $\underline{H}_k$ $\underline{n}_k$ λ and W_k represent k-DFT of h $\underline{r}$ $\underline{x}$ and w respectively then

$$R_k = \underline{H}_k Y_k + \vartheta_k$$

From which the deconvolution solution can be obtained as

$$\underline{x}_k = F_k^{-1} (R_k/\underline{H}_k) \qquad (5\ 12)$$

provided none of the component of $\underline{H}_k$ are zero

5 1 3 N ed for Deconvolution by k Circulant Completion

Note that when som of the DFT coefficients of the channel are zero (Δ) cannot be used for deconvolution as division by zero is not permitted   In such cases even if H_1 is singular for some other k H_k may oe invertible (1 all the k–DFT coefficients are nonzero)   Thus ir some cases it ma bc mor advantageous to use k DFT ins ead of 1–DFT As an illustration consider a simpl example of a channel having the following H matrix when $n = 4$ and $m = 3$

$$H = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \qquad H_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \qquad H_{-1} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

It is not diffi ult to see tha H_1 (Circulant completion) is singular whereas H_{-1} the Skew Circulant completion of H is nonsingular   In fact $\det(H_k)$ for this channel is 1–k Therefore H_k is nonsingular for $k \neq 1$

It may be further noted that computation of k–DFT is as simple as 1–DFT computation by FFT   To see this consider

k DFT of $\underline{b}$ j ven by (5 6) It can be alternatively wr_tten as

$$B_{k1} = \sum_{j=0}^{n-1} (b_j \lambda^j) \; w^{1j}$$

where λ is an nth root of ← Computation of B_{k1} s may be arried out by computing FF I of $b_j \lambda^j$ s j = 0 1 n-1 instead of b_j s Th s aspect na r s -DI T application more attractive

5 2 DeCONVOLUIION BY DIR CT I VERSIO

Th stimate x_{1e} or y_k obta ned via l DFT and k--DFT gi ven by (5 4) ano (5 12) can als be obtain d by directly inverting H_k whenev r H_k is nonsingular Th refore we can rite

$$\underline{x}_{ke} = H_k^{-1} \; r \tag{5 13}$$

ve now pre cnt two methods of inverting H_k

Me hod 1

H_k can be expressed as (see Appendix B)

$$H_k = D^{-1} \; C \; D$$

where D is a diagon l matrix with 1 λ λ^2 λ^{n-1} as the diagonal ntries C is a Circulant matrix with first column elements $D\underline{h}$ where h is the c nn l impulse response vector Therefore C^{-1} can b obtain d by using FT Then H_k^{-1} is given by

$$H_k^{-1} = D^{-1} C^{-1} D$$

when n is a power of 2 inversion of C can be carried out using $2n\log_2 n + n$ complex arithmetic operations [82] Ther fore the complexity of this method is $O(n\log_2 n)$

Elements of the first column o H_k^{-1} are obtained a follows L t $C_0\ C_1$ C_{n-1} denote first column leme ts of C^{-1} If $C_0\ C_1$ C_{n-1} denotes the first column elements of H_k^{-1} then C_1 are giv n l / $C_1 = C_1/\lambda^1$ $1 = 0\ 1$ $n-1$

Method 2

In this m thod sparse naiuie of H_k is exploitea to derive an $O(n\log_2 g_1)$ comolexity algorithm for inv rting H_k^{-1} First v illustrate the method for Circulant matrices when $n = 2g_1$ wher $g_1 = g+1$ Let $n = 8$ and $g = 3$ Then th Circulant matrix has the following form

$$H_1 = \begin{bmatrix} h_0 & 0 & 0 & 0 & 0 & h_3 & h_2 & h_1 \\ h_1 & h_0 & 0 & 0 & 0 & 0 & h_3 & h_2 \\ h_2 & h_1 & h_0 & 0 & 0 & 0 & 0 & h_3 \\ h_3 & h_2 & h_1 & h_0 & 0 & 0 & 0 & 0 \\ 0 & h_3 & h_2 & h_1 & h_0 & 0 & 0 & 0 \\ 0 & 0 & h_3 & h_2 & h_1 & h_0 & 0 & 0 \\ 0 & 0 & 0 & h_3 & h_2 & h_1 & h_0 & 0 \\ 0 & 0 & 0 & 0 & h_3 & h_2 & h_1 & h_0 \end{bmatrix} = \begin{bmatrix} P & Q \\ Q & P \end{bmatrix}$$

Note that h_1 is a sparse matrix. Direct application of FFT means comout t on of DFT of the first column having 4 zeros. Another mor efficient approach is to w ite H_1^{-1} as

$$H_1^{-1} = \begin{bmatrix} A & B \\ B & A \end{bmatrix}$$

Whenever H_1^{-1} exists $H_1^{-1}H_1 = I_n$. Thus we obtain

$$PA + QB = I_{g_1}$$

$$PB + QA = O_{g_1}$$

(5 14)

Solving the two equations w obtain

$$A+B = (P+Q)^{-1} \quad \text{and} \quad A-B = (P-Q)^{-1}$$

wh re

$$P+Q = \begin{bmatrix} h_o & h_3 & h_2 & h_1 \\ h_1 & h_o & h_3 & h_2 \\ h_2 & h_1 & h_o & h_3 \\ h_3 & h_2 & h_1 & h_o \end{bmatrix} \quad P Q = \begin{bmatrix} h_o & -h_3 & -h_2 & -h_1 \\ h_1 & h_o & h_3 & -h_2 \\ h_2 & h_1 & h_o & -h_3 \\ h_3 & h_2 & h_1 & h_o \end{bmatrix}$$

Cleail/ P+Q is a circulant while P-Q is a Skew Circulant. Note that both are full matrices (no zero entries). If g_1 is a power of 2 we can invert (P+Q) and (P-Q) using $O(n\log_2 g_1)$ conoutations

Now let us consider the general case where g_1 divides n such that $r = rg_1$ ($r > 2$). We can partition H_1 in this case as

$$H_1 = \begin{bmatrix} A_o & O & O & & O & A_1 \\ A_1 & A_o & O & & & O \\ O & A_1 & A_o & & & O \\ O & & & & & O \\ \\ O & & & & A_1 & A_o \end{bmatrix}$$

Let H_1^{-1} exist and

$$H_1^{-1} = \begin{bmatrix} B_o & B_{r-1} & & B_1 \\ B_1 & B_o & & B_2 \\ \\ B_{r-1} & B_{r-2} & & B_o \end{bmatrix}$$

Using the identity $H_1^{-1} H_1 = I_n$ we obtain

$$\begin{aligned} A_o B_o + A_1 B_{r-1} &= I_{g_1} \\ A_o B_{r-1} + A_1 B_{r-2} &= O_{g_1} \\ A_o B_{r-2} + A_1 B_{r-3} &= O_{g_1} \\ \\ A_o B_1 + A_1 B_o &= O_{g_1} \end{aligned} \qquad (5\ 15)$$

By adding the above equations we obtain

$$(B_o + B_1 + \quad B_{r-1}) = (A_o + \quad _1)^{-1} \qquad (5\ 16)$$

How consider r rth roots of unity They are

$$w^1 \quad _1 = 0\ 1 \qquad r-1 \quad \text{where} \quad w = e^{J2\ /_1}$$

Multiplying ith equation in ($_o$ 1) $0 < _1 < r-1$ by w^1 and adding we can obtain

$$(B_o + w^{r-1}B_1 + \quad wB_{r-1}) = (A_o + w^{r-1}A_1)^{-1} \qquad (5\ 17)$$

Similarly multiplying ith equation in (5 14) by w^{1k} and adding we get the general rel tionship

$$\sum_{1=o}^{r-1} B_1 w^{1k} = (A_o + w^{r-k}A_{r-1})^{-1} \quad k = 0\ 1 \qquad r-1 \qquad (5\ 18)$$

which contains (5 16) and (5 17) as particular cases

Thus in order to determine B_1 s we have to form r k Circulants (k will be different for each Circulant) whose inver ion can be achieved using $O(n\log_2 g_1)$ computations when g_1 is a power of 2 Finally B_1 s are obtain d using (5 18) with $O(g_1 \log_2 g_1)$ more computations Thus the to al complexity of the method is $O(n\log_2 g_1)$ whi h grows linearly with n for fixed g_1

Extension to the ca e of H_k^{-1} is straightforward   In this cas for $n = rg_1$ we can write

$$
H_k = \begin{bmatrix}
A_o & 0 & 0 & & 0 & k A_1 \\
A_1 & A_o & 0 & & 0 & 0 \\
0 & A_1 & A_o & & & 0 \\
& & & & & \\
0 & 0 & & & A_1 & A_o
\end{bmatrix}
$$

Since inverse of a k-C rculant is also a k-Circulant w can partition H_k^{-1} as

$$
H_k^{-1} = \begin{bmatrix}
B_o & kB_{r-1} & & kB_1 \\
B_1 & B_o & & kB_2 \\
& & & \\
B_{r-1} & B_{r-2} & & B_o
\end{bmatrix}
$$

Thus equation (5 14) becomes

$$
A_o B_o + k A_1 B_{r-1} = I_{g_1}
$$
$$
A_o B_{r-1} + A_1 B_{r-2} = 0_{g_1}
$$
$$
A_o B_1 + A_1 B_o = 0_{g_1}
$$

(5 19)

low let λ be an rth root of k   Then multiplying ith equation

in (5 19) $0 < i < r-1$ by $(\lambda w)^{iJ}$ and adding ve obtain

$$\sum_{i=0}^{r-1} B_i w^{iJ} \lambda^i = (_0 \vdash (\lambda w)^{r-J} A_i)^i \qquad J = 0\ 1\ 2 \qquad r-1$$

$$(5\ 20)$$

It may be no ed that he p ocedur is s mila to the case of

Circular t matri   inversion

5 3   DAT4 PErOVERY SCHEiLS B wED ON k-CIRCUL NT CO'PLtTIONS

When S IR is hi jh thc I -DFT pproach aescrib a in

wection 5 1 2 can be used for i cov ring data   Such a scheme

is shown in Figure 5 1   Although two FFT proce sois are shown

in the figure one is sufficient in actual practice for at a

time only one of them will bc in use

Note that in the case of the DP scheme showr in Fig 5 1

for every block of data transmitted k-DFT of the receIved

vector ani an inverse DFT i to be performed   One way of

avoiding calculation of DFf for every block separ tely is to

use the clements H_k^{-1} directly in the DR schem   Such a DR

sch m is shown in Fig 5 ?   This scheme involves a circula-

ting hift register of lencth   n in which fir t row elenents

or H_k^{-1} are stored and circulated   The tao coeffici nts of the

cyclic transversal filter ere the receIved observations

$i_0\ r_1$ r_{n-1} The multiplier with coefficient   k in the

feedback loop of the circulating rilter dep nds upon the comple

tion used
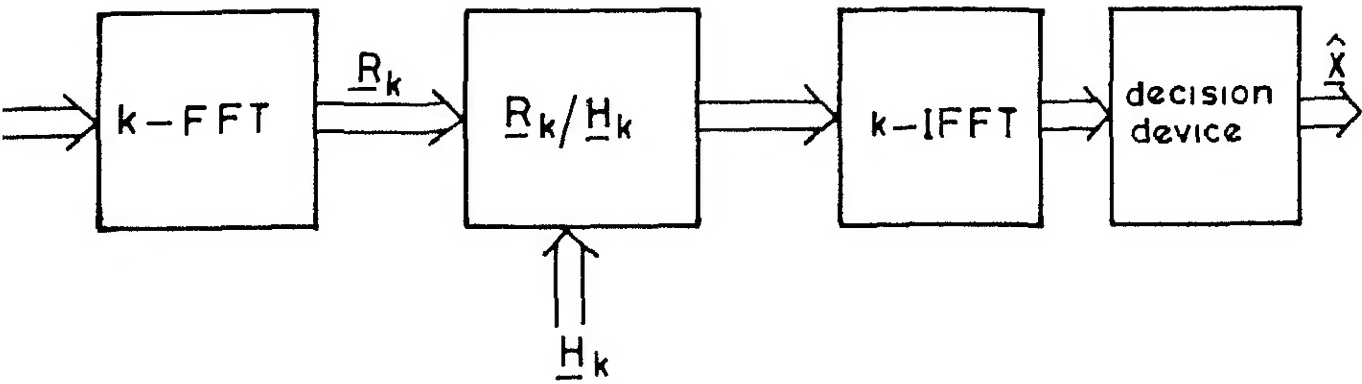
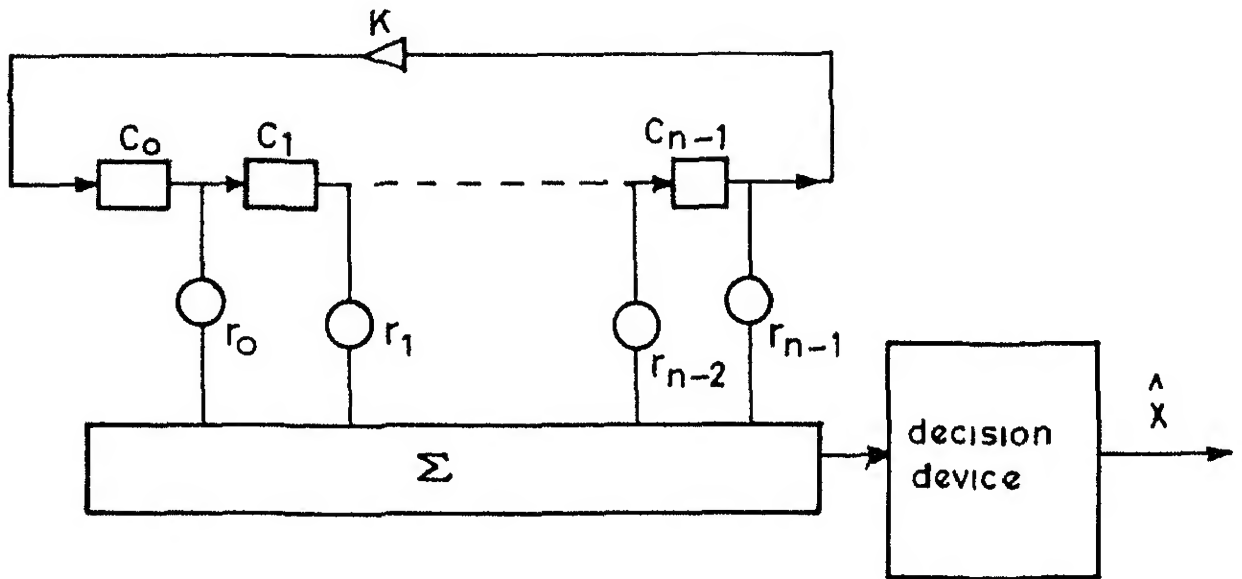FIG 5 1   Block diagram of the data recovery scheme using k-DFT

FIG 5 2   Data recovery scheme based on k-Circulant completion

Processing of each data block consists of m cycles   If
the entire processing is completed in nT seconds where T is
the symbol duration   there will be no need for long buffers to
store the incoming blocks   In this case the delay involved
will be one block dur tion   Thus two registers of length n are
sufficient   One is used in   the processing whil   the other one
stores observations arriving due to the next block

The complexity of the DP scheme shown in Fig   5 1 depends
on the method used for implementing FFT   Since FFT algorithm
is parallel and recursive it is suitable for those real time
applications where cost of the DR scheme is more important than
the probability of error performance

major advantage of the DR scheme shown in Fig   5 2 is
that in its implementation complex arithmetic is totally
avoid d   Moreover   its implementation is simple compared to
DR schemes considered in the previous chapters

5 4   PERFORMANCE CONSIDERATIONS

In the context of data transmission   it has generally
b en observed that the complexity of DR schemes seem   to be
directly proportional to the desired performance level   Thus
minimum probability of error (the best performance) demands
highly complex DR schemes   Therefor   we may infer that DR
schemes based or k-Circulart comole ion being less complex

the r performance nay not be satisfactory compared to DP
ch nes of chapt rs 3 and 4 However the Dr sch mes con-
sidered in this chapter have a distinct advantage over DR
sche1 s a veloped in th prev ous chapters when the S IR is
high   In this section we obtair explicit expressions for
the probability of error and comoare it with the performance
o DR scheme based on least square deconvolu ion   Simulation
studies on the performance of the DP schemes tudied in this
chapter are presented in Chaoter 8

To begin with let k = 1   Substituting for r fron (5 1)
in (5 1) we obtain for k = 1

$$\tilde{x}_{1e} = \underline{x}_e + H_1^{-1} \underline{w} \tag{5 21}$$

From (5 21) we s e that the effective noise variance depends
on th elements of H_1^{-1} Since the elements in any row is a
cyclically shifted ver ion of the first row and since noise
corponents are statistically independent and re of equal
variances the effective variance σ_1^2 is given by $\sigma_1^2 = \sigma_2 d_1^2$
where

$$d_1^2 = \sum_{\iota=1}^{n} h_{1\iota}^2 \tag{5 22}$$

and $\underline{h}_1$ is the first column of H_1^{-1} With th knowledge of the
DFT coeffici nts of $\underline{h}$ we can exor ss d_1^2 alternatively as

$$d_1^2 = \frac{1}{n} \sum_{\iota=1}^{n} \left(\frac{1}{\lambda_\iota \ \lambda_{n \ \iota+2}} \right) \qquad \lambda_{n+1} = \lambda_1 \tag{5 23}$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the components of $\underline{H}$ or the eigenvalues of H_1. The following theorem provides a lower bound on d_1^2.

Theorem 5.2 : d_1^2 as defined by (5.22) is lower bounded by

$$d_1^2 \geq \frac{1}{n(\sum\limits_{i=1}^{g} h_i^2)} \qquad (5.24)$$

Proof :

We know that for a nonsingular matrix A of order n

$$AA^{-1} = I_n$$

which implies $||AA^{-1}||_e = ||I||_e$ or $||I||_e \leq ||A||_e ||A^{-1}||_e$

$$(5.25)$$

where $||A||_e$ denotes the Euclidean norm of the matrix A given by

$$||A||_e = (\sum\limits_{i=1}^{n} \sum\limits_{j=1}^{n} a_{ij}^2)^{1/2}$$

Now let $A = H_1$; then $||H_1||^2 = n \sum\limits_{i=1}^{g} h_i^2$ $\qquad (5.26)$

If h_{1i} $i = 1,2, \ldots, n$ denote the elements in the first column of H_1^{-1} we have

$$||H_1^{-1}||_e^2 = n \sum\limits_{i=1}^{n} h_{1i}^2 \qquad (5.27)$$

Using (5 25) (5 26) and (5 27) v obtain

$$n < n^2 (\sum_{i=1}^{n} h_{li}^2)(\sum_{i=1}^{g} h_i^2) \quad \text{or}$$

$$\sum_{i=1}^{n} h_{li}^2 > \frac{1}{n(\sum_{i=1}^{g} h_i^2)}$$

Using (5 22) in the above inequality we obtain (5 24)

(5 22) and (5 24) are also valid in the case of k Circulant

completion since $|k| = 1$ Therefore we can write

$$d_k^2 = \sum_{i=1}^{n} |h_{Ki}|^2 > \frac{1}{n(\sum_{i=1}^{g} h_i^2)} \qquad (5\ 28)$$

It is interesting to note that all the data symbols recovered
have the same probability o being in error because SNR asso-
ciated with each of them is same   It is given by

$$(SNP)_i = \frac{1}{d_k^2 \frac{N_o}{2}} \qquad i = 0\ 1 \qquad m-1 \qquad (5\ 29)$$

When $|k| = 1$ The probability of error (PER) is given by

$$PER = \int_{\frac{1}{d_k \sqrt{(N_o/2)}}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy = Q(\frac{1}{d_k \ulcorner(N_o/2)}) \qquad (5\ 30)$$

Let us now consider th probability of rror performance of
the DR scheme based on LS-deconvolution solution given by
(5 2) In this case SNR associated with data symbols will be
different The average probaoility o error is given by

$$\overline{PEP}_{LS} = \frac{1}{m} \sum_{i=1}^{m} Q\left(\sqrt{\frac{2}{d_{Li}^2 \frac{N_o}{2}}}\right) \tag{5 31}$$

where $d_{Li}^2 \frac{N_o}{2}$ is the variance of the noise componen associated
with ith data symbol An pper bound for PER_{LS} can be obtained
as

$$PER_{LS} < Q\left(\sqrt{\frac{2}{d_{max}^2 N_o}}\right) \tag{5 32}$$

where $d_{max}^2 = \max_{1 \leq i \leq m} d_{Li}^2$

We note that d_{Li}^2 is the squared sum of the elements of the ith
row of $(H^TH)^{-1}H^T$ Equivalently they are the diagonal elements
of $(H^TH)^{-1}$ (because $(H^TH)^{-1} H^TH(H^TH)^{-1} = (H^TH)^{-1}$) We now
show that

$$d_k^2 \geq d_{Li}^2 \qquad i = 1 2 \qquad m \tag{5 33}$$

os a particular case of a more general result Lct
$H_g = [h \; H_c]$ where Hg is the souare complction of H such that
H_g is invertible (Sucn conpletions are treated in greater
d tail in Chapter 6)

Let d^2_{gi} $i = 1\ 2$ n be the diagonal elements of $(H_g^T\ H_g)^{-1}$ In the following theorem we prov nat

$$d^2_{gi} \geq d^2_{Li} \tag{5 34}$$

It is easy to see that (5 33) follows as a special case of (5 34) when $|k| = 1$

Iheorem 5 3 Given H_g d_{gi} d_{Li} as defined above

$$d^2_{gi} \geq d^2_{Li} \qquad i = 1\ 2 \qquad m$$

Proof $(H_g^T\ H_g)^{-1}$ can be written as

$$(H_g^T\ H_g)^{-1} = \begin{bmatrix} H^T H & H^T H_c \\ H_c^T\ H & H_c^T\ H_c \end{bmatrix}^{-1} = \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix}$$

Since

$$\begin{bmatrix} H^T H & H^T H_c \\ H_c^T H & H_c^T H_c \end{bmatrix} \begin{bmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{bmatrix} = \begin{bmatrix} I_m & O \\ O & I_{n-m} \end{bmatrix}$$

We have

$$H^T H\ G_{11} + H^T H_c G_{21} = I_m \tag{5 35}$$

$$H^T H\ G_{12} + H^T H_c G_{22} = O_{mxn-m} \tag{5 36}$$

Therefore from (5 36) w obtain

$$H^T H_c = -H^T H G_{21} G_{22}^{-1}$$

where G_{22}^{-1} exists becau e $H_g^T H_g$ is a positive definite matrix

substituting for $H^T H_c$ in (35) w get

$$(H^T h)^{-1} = G_{11} - G_{12}G_{22}^{-1}G_{21} \qquad (5\ 37)$$

$G_{21} = G_{12}^T$ a $H_g^T H_g$ is a symm tric matrix

Now let $\tilde{d}_1^2$ be the diagonal el ments or $G_{12}G_{22}^{-1}G_{21}$

Since $G_{21} = G_{12}^T$ and G_{22} is a p d m tri

$$G_{12}G_{22}^{-1}G_{21} > 0 \text{ (by the definition or a p d matrix A}$$
x^T $\underline{x} > 0$ for all $\underline{x} \in R^n$ xcept $x = o$

which shows that $\tilde{d}_1^2$ is strictly positive Now using (5 37)

$$d_{L1}^2 = d_{g1}^2 \quad \tilde{d}_1^2$$

where d_{g1}^2 are the diagonal entries of G_{11}

**

Summarizing in this chapter we have s udied the applica-
tion of k Circulant compl ctions for data reco/ r/ in the con-
text of BDT We h ave analysed two DR schemes which facilitate
good reduction in omputational complex t/ and torage require-
ents compared to the DR sche es studied in the previous
chapter Moreover these schemes are am nable to performance
analysis W have obtained explicit expressions for the proba-
bility of rror and compar d it with th p rformanc achievabl by

DR scheme based on LS-decorvolution   In general   the perfor-
mance of DR schemes based on k Circulant completions is infe-
rior to the DR schemes of chapters 3 and 4   However   in
Chapter 8 we   shall see that   in the case of some channels
DR schemes based on k-Circulant compl tion   perform well
compared to ML-DR and DF-DP schemes

In the next chapter w   take up a detailed study of the
relationship between least squares deconvolution and deconvo-
lution via k-Circulant completions

But $(H^TH)^{-1} H^T = \frac{1}{3} \begin{bmatrix} 2 & 1 & -1 \\ -1 & 1 & 2 \end{bmatrix}$

Ther fore $\underline{x}_{LS} = \begin{bmatrix} 1 & 1\smile \\ -1 & 23 \end{bmatrix} \neq \begin{bmatrix} 1 & 6 \\ -1 & 1 \end{bmatrix} = x_1$

Thus it becomes impo tant and interesting to know condi-
tions under wh cl $x_{L\smile} = {}_k$ b caus in su r a case LS solu-
tion can be obtaine l wit1 much less computation via k Circulant
completions

In the present chart r we study this problem in a more
g n ral framework of square completion of rectangular matrices
Let $H_s = [H \quad H_{s1}]$ be nonsingular square compl ction of H
in ge1 ral H_{s1} need not depend on H T1e k Circulant comple-
tion of H is a particular case of H_s wh re H_{s1} depends on H
Let $H_s^{-1} = \begin{bmatrix} H_{s2} \\ -H_{s3} \end{bmatrix}$ Our interest is to obtain conditions under-
which

$$x_{LS} = x_s = H_{s2} \ \underline{r} \qquad\qquad (6\ 3)$$

/e how that $\underline{x}_{L\smile} = \underline{x}_s$ if and only if $r^TH_{s1} = 0_{mxn-m}$ Such
completions are called or hogonal co pletions As a corro-
llary it follows that $x_L = \underline{x}_k$ iff $H^TH_{\kappa 1} = 0_{mxn-m}$ Under
this condition we fur h r show that orthogonal k-Circulant
completions of H are orthogonal k Circul nt matrices

We organize the chapter in the following manner   In Section 6 1 a method of solving overdetermin d set of linear equations using the notion of squar conpletior of natrices is discussed   A brief review of least squares solution of lin ar system of equations and Moor -Penrose (MP) inverse of matric s is given in Section 6 2   In section 6 3 we obtain conditions on squar completions that result in P-inv rse or least squares inverse (LS-inverse) of a full ank rectangular matrix Squar completions which result in MP-inverse are called ortho gonal square completions   W further sho that H_k is an ortho gonal k Circulant completion of H iff H_l is an orthogonal k- Circulant   In Section 6 4 we determine channels which are amenable to LS solution via k-Circulant completions   We show that in most of the practic l cases LS solution to the deconvo- lution problem cannot be obta ned via (6 2) because in such cases k-Circulant completions do not result in orthogonal k- Circulants   In Section 6 5 we characteriz orthogonal k- Circulants in the k-DFT domain   An alternat expre sion for LS solution using partitions of k-Circulant completions of H is obtained in Section 6 6 for the purpo e of knowing the relationship between H_{k2} and th MP-inver   of H   We show that

$$H^- = H_{k2} - H_{k2} F_k (H_{k3} H_{k3}^r)^{-1} H_{k3}$$

wher H is the Moore-Penros inverse of H

6 1 SQUARE COMPLETION AND SOLUTION OF LINEAR EQUATION

s the deconvolution problem is essentially solving a s t
of linear equations we first consider application of square
completions for solving a system of linear equations given by

$$A \underline{x} = b \tag{6 5}$$

where A is an nxm matrix with n > m Let rank of $A(r(A))$ be m
Let $A_s = [A \quad A_{s1}]$ be a nonsingular square completion of A
Such a choice of A_{s1} is possible because A is a full rank
matrix Let $\underline{x}$ be an extended vector

$$y_e = \begin{bmatrix} x \\ 0 \end{bmatrix} \tag{6 6}$$

where 0 is an n-m length zero vector In terms of A_s^{-1} we can
obtain a solution to (6 5) given by

$$\tilde{x}_e = A_s^{-1} \underline{b} \tag{6 7}$$

Note that $\tilde{\underline{x}}_e \neq \underline{x}_e$ in general We will return this point
later Now let

$$\tilde{x}_e = \begin{bmatrix} \underline{x}_1 \\ x_2 \end{bmatrix} \tag{6 8}$$

where x_1 is a solution of (6 5) with respect to the comple-
tion A_{s1} Depending on the choice of A_{s1} different solutions
are obtained natural question then demands answer at this

stage is wheth r all such solu ions ar equivalent in so e
sense or not Because if th solution given by (6 6) i
ind p ndent of A_{s1} it may be economical in terms of computa-
tions to choos A_{s1} such tha inversion of $_s$ involves least
omputational complexi y One simple resul in this connect on
is th following

Theorem 5 1 If $\underline{b}$ belongs to th range space of A then $\tilde{\underline{x}}_e$
giv n by (6 7) is indepenc nc of A_{s1}

Proof L t $\underline{a}_0$ $\underline{a}_1$ $\underline{a}_{m-1}$ be the columns of A Since $\underline{b}$
is in the range space of A $b = c_0 \underline{a}_0 + c_1 \underline{a}_1 + \quad c_{n-1} \underline{a}_{m-1}$ for
some c_i s Therefore solution to (6) is given by

$$\underline{x} = (c_0 \; c_1 \qquad c_{n-1})^T = \underline{c}$$

L t A_s be any nonsingular square completion of A Let
$$_s^{-1} = \begin{bmatrix} A_{s2} \\ \bar{A}_{s3} \end{bmatrix}$$ Then from (6 8) we see that

$$\underline{x} = A_{s2} \, \underline{b} = \quad_{s2} \quad \underline{c}$$

Since $_s^{-1} A_s = I_n$ we obtain $A_{s2} A = I_m$ Therefore $\underline{x}_1 = \underline{c}$
which is independent of $_2$ **

Note that the condition on b demanded by Theorem 5 1
cannot be guaranteed in practic i b does not belong to
the range space of A We consider this case in S ction 6 3
In th next section we give a brief review of the conventional
methods available for d aling with this case

6 2   LEAST SQUARE MINIMUM NORM SOLUTION AND MOORE PENROSE GENERALIZED INVERSE

The method of solution normally used when $\underline{b}$ is not in the range space of A is to constrain the solution space   The method of least squares is one way of constraining the solution space   We choose only those x s which minimize

$$\| x - \underline{b} \|^2 = (x   b)^T (A\underline{x} - b) \tag{6 9}$$

when A is of rank m it can be shown that the right hand side of (6 9) is minimized by

$$x_L = (A^T A)^{-1} A^T b$$

when   does not have full rank ($r(A) < m$) the method used is to further constrain the solution space by considering $\underline{x}$ s which minimiz (6 9) and have minimum length   It can be shown that such a solution is unique and is given by [85] [87]

$$\underline{x}_{mLS} = A^+ \underline{b}$$

where A^+ is called the Moore-Penrose (MP) inverse of A[85] A^+ satisfies the following properties

1   AA^+ $= A$
2   A^+AA^+ $= A^+$
3   (A^+) $= AA^+$
4    $(A^+A)^*$ $= A^+A$

$$\tag{6 10}$$

A matrix A^{LS} which satisfies the conditions $AA^{LS}A = A$ and $(AA^{LS}) = AA^{LS}$ is called a least square inverse of [8]. When A is of full rank it can be shown that $A^{LS} = A^+$

3 LEAST SQUARES SOLUTION VIA SQUARE COMPLETIONS

Another approach that can be taken when b is not in the range space of is to extend the solution space. It is easy to see that square completion is a way of extending the solution space. The solution obtained by forming $A_s^{-1}b$ depends on A_{s1} which controls the solution space. In such situations it is instructive to know what kind of completions lead to least squares solution. Alternatively we are interested in A_{s1} s which make A_{s2} in $A_s^{-1} = \begin{bmatrix} A_{s2} \\ s3 \end{bmatrix}$ a least squares inverse or Moore-Penrose inverse. The following lemma characterizes MP inverse of A via square completions.

Lemma 6 1 Let $A_s = [A \quad s1]$ and $A_s^{-1} = \begin{bmatrix} A_{s2} \\ A_{s3} \end{bmatrix}$. Then A_{s2} is the MP inverse of A iff $(A_{s1}A_{s3})^k = {}_{s1}A_{s3}$

Proof We know that $A_s^{-1}A_s = A_sA_s^{-1} = I_n$. Therefore in terms of the partitions of A_s and A_s^{-1} the following matrix equations must be satisfied

$$_{s2} \qquad = I_n \qquad\qquad (6\ 11a)$$

$$_{s2}\ _{s1} = O_{m \times n-m} \qquad\qquad (6\ 11b)$$

$$_{s3} \qquad - O_{n-m \times m} \qquad\qquad (6\ 11c)$$

$$_s\ {}_{s1} = I_{n-m} \qquad\qquad (6\ 11d)$$

$$_{s2} \quad {}_{s1}A_{s3} = I_n \qquad\qquad (6\ 11e)$$

Using (6 11a) it is easy to show   t A_{s2} satis ies the first
three properties of he P-inverse of A given b/ (6 10)   The
 ou th prop rty in (6 10) requires t at

$$(\quad_{s2}) = \quad_{s2}$$

From (6 11e) we ca se that this condition vill b satisfied
iff $(\quad_{s1} \quad_{s3}) = \quad_{s1} \quad_3$

*

Using L mma 6 1 we can prove the following t eore

Th orem 6 2   Gi en a nonsingular comnl 10 of A
$A_s = [\quad A_{s1}]$ and it inverse $\quad_s^{1} - \begin{bmatrix} A_{s2} \\ A_{s} \end{bmatrix} \quad_{s2}$ 1 th P
invers of   iff $A^T A_{s1} = O_{nxn-m}$

P oof   Let A_{s2} be   P inv rse of A   Then $_{s2} = (A^T A)^{-1} A^T$
 ubstituti g for A_{s2} in (c 11b) wc see that $A^T A_{s1} = O_{mxn-m}$
This proves the first part of the theorem

  Now assume that $A^T A_{s1} = O_{mxn-m}$ Let (A) denote the
r nge pace of A   Then $R(A) \perp R(A_{s1})$ Here $\perp$ indicates that
$R(A_{s1})$ is orthogonal to $R(A)$ From (6 11b) we have
$P(A_{s2}^T) \perp R(A_{s1})$ Since $P(A_{s1}) \perp R(A)$ it follo s that the rows
of A_{s2} pans $R(A)$ But columns of A also sp n $P(A)$ So
$A_{s2} - DA^T$ for some nonsin jular matrix D   Subs t tuting for A_2
in (6 11) we get $D = (A^T A)^{-1}$ Tha implies $A_{s2} = (A^T A)^{-1} {}^T$

**

  From now onward we shall r fer to square completions
s
having th property $\bar{A} A_{s1} = O_{mxn-m}$ as orthogonal completions

In th nex+ th orem v consider the impli ⟶tior o Theorem 6 1 in the ca e of k-Circulant completions

eoren 6 3   Giv n a nonsingular k-Circulant co pletion of H

$H_k = [H \ H_{k1}]$ and its inverse $H_k^{-1} = \begin{bmatrix} H_{k2} \\ H_{k3} \end{bmatrix}$ $_{12}$ is the P-

in/ rse of H iff H_k 1 an oi thogonal k-Circulant atrix

Proof   From the definition o k-Circulant atrix (Appendix B) v   c 1l that h_1 th irst column of H_1 is related to oth r columns or H_k through η_k which is a k Circulant matrix with first ro/ (0 0   k)   n other words

$$h_i = \eta_k^{i-1} \underline{h}_1 \quad i = 2\ 3 \qquad n \qquad\qquad (\text{o } 12)$$

wh rc $\underline{h}_i$ is th ith colum of H_k

Using Theor m 6 2 w~ know that H_{k2} 1 the MP inverse of H if $H^T H_{k1} = O_{n \times n-m}$

For this condition to hold ve must hav

$$\underline{h}_j^T \underline{h}_i = 0 \qquad j = 1\ 2 \qquad m \quad i = i\!+\!1 \qquad n$$
$$(6\ 13)$$

In view of (6 12) (o 13) implies

$$h_1^T (\eta_k^{j-1})^T \eta_k^{i-1} \underline{h}_1 = 0 \qquad j = 1\ 2 \qquad m$$
$$i = m+1 \qquad n$$

But $\eta_k^T \eta_k = k^2 I_n$ (see Appendix B)

Therefore we obtain $\underline{h}_1^T \eta_1^i \, {}^J h_1 = 0$

or $\eta_1^T \eta_k^i h_1 = 0$ $\quad i = 1\ 2 \quad\quad n-1$

But $\eta_k^i \underline{h}_1$ is the $i+1$th column of H_k

Therefor H_k in this case is an orthogonal k-Circulant

From the foregoing discussion it is clear that the
square completion of rectangular matrices provide a general
framework for the study of deconvolution problems Both
the least squares approach and the method based on k-Circulant
completion are particular cases with appropriate choice of the
completions In the light of Theorem 6 3 in the next section
we determine those channels which admit LS solution via k-
Circulant completions

6 4 CHANNELS AMENABLE TO LS SOLUTION VIA k CIRCULANT COMPLETIONS

In this section we are interested in determining those
channel convolution matrices H such that H_k is an orthogonal
k-Circulant Two cases are to be considered

Case 1 $\quad m > g$

In the case of block data transmission $m > g$ This is
necessary because otherwise reduction in the effective data
rate over the channel will be large The form of H_k in this
case is shown below

$$H_k = \begin{bmatrix} h_o & O & O & kh_g & kh_{g\,1} & & kh_1 \\ \eta_1 & h_o & & O & kh_g & & kh_2 \\ & h_1 & & O & \cup & & \\ & & 1_o & & & & \\ h_g & & h_1 & h_o & & & kh_g \\ O & h_g & & h_1 & h_o & & O \\ & O & & & & & \\ & & & & & & \\ O & O & h_g & h_{g-1} & h_{g\,2} & & h_1 & h_o \end{bmatrix}$$

I is easy to see that for H_k to be n orthogonal k-Circulant the following equations are to be satl fied

$$\nvdash h_o h_g = O$$

$$k(h_o h_{g-1} + h_1 h_g) = O$$

$$k(h_o h_1 + h_1 h_2 + \quad h_{g-1} h_g) = O$$

Since $h_o \neq 0$ we obtain

$$h_g = O \quad h_{g-1} = O \quad h_1 = O$$

This means that the only channel that admits LS solution via k-Circulant completion is the distortionless channel having $h_o \neq O$ and all other ISI components zero

I now consid r the case n > m+g   This case aris s in
most of the deconvolution applications   For instance   when
FFT is to be used for inverting H   It is desirable to hav n

a power of 2 in such situations   Therefore   wh r n is not
a power of 2   the channel vector is pa ded with zeros   Another
motivation for consid ring this case is that   ere nay exist
some n > m+g such th t LS solution is obtained via k-Circulant
completion

In this case H_k ill ha e a more general form as shown
b low

$$H_{ke} = \begin{bmatrix} H & H_{k4} \\ & - & - \\ O & H_{k \supset} \end{bmatrix}$$

where O is a null matrix of order n m-g×m   Using the arguments
given earlier we can easily show that H_{ke} is an orthogonal k
Circular t only when $h_o \neq 0$ and $h_1 = 0$ 1 = 1 2   g

Case 2   m < g

This situation arises in   convolution si uations where
the number of unknowns is less than the number of impulse res-
ponse coefficients

In this case H_k takes the follo ing form when g = 4 and
m = 2

$$H_k = \begin{bmatrix} h_0 & 0 & kh_4 & kh_3 & kh_2 & kh_1 \\ h_1 & h_0 & 0 & kh & kh_3 & kh_2 \\ h_2 & h_1 & h_0 & 0 & kh_4 & kh_3 \\ h_3 & h_2 & h_0 & h_0 & 0 & kh_4 \\ h_4 & h_3 & h_2 & h_1 & h_0 & 0 \\ 0 & h_4 & h_3 & h_2 & h_1 & h_0 \end{bmatrix}$$

For H_1 to be an orthogonal k-Circulant must have

$$kh_0 h + h_2 h_0 + h_3 h_1 + h_4 h_2 = 0$$

$$kh_0 h_3 + kh_1 h_4 + h_3 h_0 + h_4 h_1 = 0$$

$$kh_0 h_1 + kh_1 h_2 + kh_2 h_0 + h_3 h_4 = 0$$

Any solution to the above set of equations serves as the desired channel Thus for $m < g$ we see that it may be possible to obtain LS solution via k-Circulant completion Explicit form of h_1 s is difficult to obtain in this case in the next section we give an alternative characterization of orthogonal k Circulant which helps in obtaining some insight into the nature of the possible solutions

6 5 CHARACTERIZATION OF ORTHOGONAL k CIRCULANTS

From Appendix b we have

$$H_k = F_k^{-1} \, di \, g (\, \lambda_0 \quad \lambda_1 \qquad \lambda_{n-1}) \, F_k = F_k^{-1} \, \Lambda \, F_k \qquad (6\ 14)$$

and $\Gamma_k = FD$ where $D = \text{diag}(1\ p\ p^2 \qquad p^{n-1})$

Using these in

$$H_k H_k = I_n \qquad\qquad (6\ 15)$$

we obtain

$$F^t \ \Lambda \ F_k^{1^*} \ \Lambda F_k = I_r$$

or $$D\ F^t \Lambda^{*}{}^{-1} D^1\ D^1 F^{-1} \Lambda\ \Gamma D = I_n$$

where $\Lambda = \text{diag}(\lambda_o\ \lambda_1 \qquad \lambda_{n-1})$

Noting that $F = \Gamma^{-1}$ we obtain

$$\Lambda\ F\ (DD^*)^{-1} F^1 \ \Lambda\ F = D^{*^{-1}} D^{-1} \qquad\qquad (6\ 16)$$

or $$\Lambda^* F(DD^*)^{-1} F^{-1} \Lambda = F(DD^*)^{-1} F^*$$

Let $$X = F(DD^*)^{-1} F^{-1} \qquad\qquad (6\ 17)$$

Then $$\Lambda^* X\ \Lambda = I \qquad\qquad (6\ 18)$$

The problem of determining orthogonal k Circulant thus reduces to he determination of Λ such that (6 18) is satisfied for the given k   If none of the elements of I are zero due to the diagonal nature of Λ it is easy o see that λ_1 s must satisfy th condi ion

$$\begin{bmatrix} \lambda_o \\ \lambda_1 \\ \\ \lambda_{n-1} \end{bmatrix} [\lambda_0\ \lambda_1 \qquad \lambda_{n-1}\ ]$$

$$= C \text{ (matrix of all ones)} \qquad\qquad (6\ 19)$$

To see this let n = 2   Then   will be of the form

$$X = \begin{bmatrix} x_{11} & 12 \\ x_{21} & x_{22} \end{bmatrix}$$

then $\Lambda X \Lambda$ =

$$\begin{bmatrix} \lambda_1^2\, x_{11} & \lambda_1 \lambda_2\, x_{12} \\ \lambda_2\, \lambda_1 x_{21} & \lambda_2^2\, x_{22} \end{bmatrix}$$

In order to satisfy (6 18) we must have

$$\lambda_1^2 = 1 \qquad \lambda_1 \lambda_2 = 1 \quad \text{and} \quad \lambda_2^2 = 1$$

The procedure for obtaining an orthojon l k-Circulant is as follows   lor the given v lue of k (6 18) mu t be solveu to obtain λ_1 s which in turn proauce H_k via '6 14)   As an illustration we con ider he case when l = 1

When k = 1 $(DD^T)^{-1}$ becomes an identity matrix   Hence

$$X = \begin{bmatrix} 1 & 0 & 0 & & 0 \\ 0 & 0 & 0 & & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & & 0 \end{bmatrix}$$

U ing (6 18) we s e tha λ_1 s for an or hogonal circulant satisfy

$$\lambda_1^2 = 1 \qquad \lambda_2 \lambda_n = 1 \qquad \lambda_3 \lambda_{n-1} = 1 \qquad \qquad \lambda_n \lambda_2 = 1$$

For a real matrix H_1 w have

$$\lambda_1 = \lambda_{n-1+2}^* \qquad 1 = 2\ 3 \qquad\qquad n \qquad \text{with } \lambda_{n+1} = \lambda_1$$

Therefore

$$|\lambda_1|^2 = 1 \quad 1 = 1\ 2 \qquad\qquad \text{M} \quad \text{where } 1 = \begin{cases} \dfrac{n}{2} + 1 & \text{ir } n \text{ is even} \\[2mm] \dfrac{n+1}{2} & \text{ir } n \text{ is odd} \end{cases}$$

Different choices for $\lambda_1 \lambda_2 \qquad \lambda_1$ decice different orthogonal
real circulants   Thus we have infinit numb r of orthogonal
Circulants   The λ_1 s are given by

$$\lambda_1 = \cos\theta_1 + j\ \sin\theta_1 \qquad 0 \le \theta_1 < 2\pi$$

For instance when $n = 4$ we can have

$$\lambda_1 = 1 \quad \lambda_3 = 1 \quad \lambda_2 = \cos\theta + j\sin\theta$$

Then the first row of the desired k-Circulant is given by

$$h_1^T = (1+\cos 0 \qquad \sin\theta \qquad 1-\cos\theta \qquad \sin\theta)$$

6 6 LS SOLUTION USING k-CIRCULANT COMPLETION

From the analysis in the previous sectio1s ve conclude
that for most cases of practical interest LS solution to the
deconvolution pioblem cannot be obtained via k-Circulant comple-
tions as given by (6 1)   However although he solution given
by (6 1) is not the LS solution it may be very lose to the LS
solution   In this section we obtain an alternate exprcssion for
the LS solution using k-Circulant completions

Consider the received vector r given by

$$r = Hx + w \qquad (6\ 20)$$

We first obtain the LS solution using heuristic arguments   If w w re to be a zero vector and H_k^{-1} exists for some k   we can obtain exact solution to (6 20) as

$$x = x_1 = H_{k2}\ r \quad \text{and} \quad \underline{x}_2 = H_{k3}\ r = 0 \qquad (6\ 21)$$

wher

$$H_k = [H \quad H_{k1}] \qquad H_k^{-1} = \begin{bmatrix} H_{k2} \\ k3 \end{bmatrix} \qquad x_e = \begin{bmatrix} x \\ 0 \end{bmatrix} = \begin{bmatrix} x_1 \\ \underline{x}_2 \end{bmatrix}$$

On the otherhand   when $\underline{w}$ is not a zero vector we have in general $x \neq x_1$ and $\underline{x}_2 \neq 0$ Now con id r x_2 given by $\underline{x}_2 = H_{k3}\ \underline{r} = H_{k3}(Hx + w)$ But $H_{k3}H = 0_{n-m\times m}$ Therefore

$$H_{k3}\ \underline{r} = H_{k3} \qquad (6\ 22)$$

The nimum no m l ast square solution (6 22) is given by

$$\hat{w} = H_{k3}(H_{k3}\ H_{k3}^{*})^{-1}\ H_{k3}\ \underline{r} \qquad (6\ 23)$$

Thus we have estimated the noise vector $\underline{w}$ in the best possible manner under th given conditions   Ve can use this estimate to reduce the o fect of noise in (6 20)   Substituting for $\underline{r}$ using (6 20) in (6 21) for x_1 we obtain

$$\underline{x}_1 - H_{k2}\ r = \underline{x} + H_{k2}\ \underline{w}$$

Using (6 2∂) for $\underline{w}$ we get

$$H_{k2} \; r = x + H_{k2} \; H_{k3}(H_{k3} \; H_{l3}^{*})^{-1} \; H_{k3} \; \underline{r}$$

or $\quad \underline{x} = H_{k2}[I - H_{k∂}^{*}(H_{k3} \; H_{k3})^{-1} \; H_{k∂}] \; r$

Ic now claim that no b tter result can be achieved under th given conditions   A formal proof of his claim is given b low

Tl orem 6 4 $\quad P^{+} = H_{k2}[I-H_{k3}(H_{k} \quad H_{k∂}^{*})^{1} \; H_{k3}]$ $\qquad$ (6 24)

Proof   Ie show that (6 24) sat sfies ll the four properties of tne MP-inverse given by (6 10) Using (6 11) whe e ins ead of A_s we have $\quad H_k$ and H_l^{-1} replace A_s^{-1} with $H_{k1} = A_1$ $H_{k2} = A_{s2}$ and $H_{k3} = A_{s3}$ it is ea sy to show that representation of P^{+} given by (6 24) satisfies th followin∩ condit ons

$$HH^{+}H = H$$

$$H^{+}HH^{+} = H^{+}$$

and $\qquad (H^{+}H)^{t} = H^{+}H$

Ie hav to finally show that $(HH^{+})^{*} = HH^{+}$ For this we make use of (6 11e) which gives after appropriate subst tutions

$$HH_{k2} + H_{k1}H_{k3} = I_n$$

Therefore $\quad HH^{+} = HH_{k2}[I-H_{k3}^{*}(H_{k3}H_{k3}^{t})^{-1} \; H_{k3}]$

$$= [I-H_{k1}H_{k3}][I-H_{k3}(H_{k3}H_{k3}^{t})^{-1} \; H_{k3}]$$

$$= [I - H_{k3} (H_{k3} H_{k3})^{-1} {}_{k3}] - {}^{-1}{}_{1} {}_{1} H_{k3} + {}^{+}{}_{1} H_{k3}$$

$$= [I - H_{k}^{*} (H_{k3} H_{k3})^{-1} H_{k3}]$$

which is a symmetric matrix

therefore $(HH^{+}) = HH^{\top}$

But such a matrix is unique [35]

In the foregoing discussion we used heuristic arguments
for obtaining (6 24) Now we will show that (6 24) can b
derived in a more formal manner For this purpose we write
(6 20) using k-Circulant completion of H and extended vector x_c
as

$$\underline{r} = H_k x_e + w \qquad (6\ 25)$$

Note that $\underline{x}_e$ is an extended version of x obtained by
padding $\underline{x}$ with zeros However while seeking solution to
(6 5) via H_k^{-1} we ignored this constraint on x_c Therefore
the solution naturally differs from x_{LS} except in some
special cases This means that in order to obtain LS solution
to (6 25) we must minimize

$$(\underline{r} - H_k x_e)^{x} (\underline{r} - H_k \underline{x}_c) \qquad (6\ 26)$$

subject to the constraint

$$C x_e = 0 \qquad (6\ 27)$$

where $C = [O_{n-m \times n-g} \quad I_g]$

This minimization problem can be solved using Lagrange multiplier method. Let $\underline{\alpha} = (\alpha_1 \; \alpha_2 \quad \alpha_g)^T$ be the multipliers Then e form

$$\Gamma(x \quad \alpha) = (\underline{r}-H_k \; x_c) \; (r-H_k \; \underline{x}_e) + 2\alpha^T \; C \; \underline{x}$$

Differentiating $F(\underline{x} \quad \alpha)$ with respect to λ and α we obtain conditions for minimum as

$$H_k^* \; H_k \; \underline{x}_c \quad H_k^* \; \underline{r} \quad C^T \; \alpha = 0$$

and $\qquad C \; x_e = 0$

As H_k is invertible we obtain

$$\underline{x}_e = H_k^{-1} \; \underline{r} - H_k^{-1} \; H_k^{*-1} \; C^T \; \alpha$$

and $\qquad \underline{\alpha} = (C \; H_k^{-1} \; H_k^{*-1} \; C^T)^{-1} \; C \; H_k^{-1} \; \underline{r}$

Existence of $(CH_k^{-1}H_k^{*-1}C^T)^{-1}$ follows by noting that

$$CH_k^{-1} = H_{k3} \quad \text{where} \quad H_k^1 = \begin{bmatrix} H_{k2} \\ H_{k3} \end{bmatrix}$$

and H_{k3} is a full rank matrix Thus

$$\underline{x}_e = H_k^{-1} \; \underline{r} - H_k^{-1} \; H_k^{*-1} C^T \; (H_{k3} \; H_{k3}^*)^{-1} \; H_{k3} \; \iota$$

Therefore the solution to the original problem is given by

$$\underline{x} = H_{k2} \; \underline{r} - H_{k2} \; H_{k3}^* \; (H_{k3} \; H_{k3}^*)^{-1} \; H_{k3} \; r$$

Using Theorem 6 4 we see that x is the LS solution to th deconvolution problem

Using (6 24) we obtain an xpression for the dif er nc $\Delta H = H_{k2} - H^+$ given by

$$\Delta H = H_{k2} \; H^*_{k3} \; (H_{k3} \; H_{k3})^{-1} \; + \;_{\cup} \qquad (6\ 28)$$

Note that ΔH is ind pendent of H^+ Therefore it c n be computed by knowing H_k^1

If the l ments o H are small the solutions obtained using h_{k2} and H^+ will not differ s gnificantly In th case of orthogonal k-Circul nt completions of H it is easy to see that $\Delta H = O_{mxn}$ as $H_{k2} \; H_{k3} = O_{mxn-m}$

In summary we hav studi d LS d c nvolution problem in the framework of squar completion of rectangular matrices We have shown that in most of the practic l situations LS solution cannot be obtained via dir ct inv rsion of k Circulant completion of the chann l convolution natrix

Another method of obtaining LS solution is by using it rative schemes In the next chapter we study steepest descent algorithm in th discrete Fourier transform donain which provides LS solution to the deconvolu ion problem

CHAPTER 7

LEAST SQUARES DECONVOLUTION - ITERATIVE APPROACH

In the previou chapter we consiuered th pos ibility of
applying computat onally attractive k-Circulant comoletions of
th ch nnel matri H for obtaining L solution to the de onvo-
lu ion p oolem which involves inversion of $H^T H$ Our analysis
shoved that in most of the practical ases the solution given
by (6 1) usin 1-Circulant comoletions do not yield the IS
solution to the deconvolution problem n alternate expression
for L solution was also obca ne in S ction 6 6 Though th
derivation of this expression mak s use of k-Circulant comple-
tions the final expr ssion is not computa ionally attractive
compared to the urect inversion of $H^T H$

In this chapter we study ano her metho of ootaining LS
solution to the ueconvolution problem It is based on the
application of wellknown steepest descent algorithm (SDA) [89]
which is an iterative technique in contrast to the noniterative
or urect approach consideied in Chapter 5

Gersho [24] appears to have used ixed step SDA for the
first tim for the purpos of adaptive channel eou lization in
which the tap oefficients of the equalizer are adjusted conti-
nuously so that the mean square error between the actual and

stimated data symbols is minimized   He has shown that in the
or sence of noise the algorithm yields a suboptimal setting of
the tap coefficints

Walzman and Schwartz [21] [2_] have studied fixed step
sup in the DFT domain for the pu pose of a tom ti channel
eoualization   Usin two examples they have observed that th
DFT domain SDA conv rge faster than the sample domain SDA as
exact Inowledg of th eigenvalu   are avail ole in th case of
DFT domain SDA   Their work has be n extended by Picchi and
Prati [64] who hav includ d self orthogonal zing feature into
the DFT do on i ration scheme   Using simulation studies th y
have shown tha the per ornance o self-orthogonalizing DFT
domain qualization scheme is sup r or to that of the sample
domain algori thm cu to Godard [63]

The observations mad by Jalzman and Schwartz [21] and
Picchi and Prati [64] ar more carefully ex nined in this
chapter   One re son for this is based on our und rstanding
that DFT b ing an orthonormal transformacion th it ra ive
sch nes in both the domains must have same convergence proper-
ties   But there i ay be co putational advantages in going rom
sample domain to th DFT a nain due to th availability of fast
algorithms for computing DFT

We devot a iajor portion of this chapter towards a b tter
understanding of the role of DFT and k-Circulant completions in

reducing processing t a and providing fast convergnc   Our
analy is clearly shows how DFT helps in reducing conputaions
Jsing th frmework o l-Circular t comolctions w show that
dvantge in tr or fster convergnc rt can bc chi v a
in the DFT domain bcau it povides in th cse o scme
xamples good stirate of the converg ic arafter it 1
ho m th t th cstinate obtained using k-Circular t c molction
can also be used in the sample dom in to obtain s ne conv r-
g nc ra e

Th chapter is orgaiz d in the followirj mann r A ori f
r vi v of steepest d cent algorithn is given in ection 7 1
Ihe nne n which DFI helps in r ducinc computat ons is dis-
cuss d in S ction 7 2 In ection 7 3 we aorive SDA in th
DFT domain from thre c f i nt vi wpoints dat recover/
schenc basca on DFT doma n SD i is iv n in ocction 7 3 1
Converg nce oropertics of SDA r discuss d in Sc tion 7 4
method of increasing the onvera nce rate of the SD is dis
cus d n Section 7 5 le sho ho/ Circulant or k-Circulan
comolo ions c n be usca to r duc numbei or iter tions This
analysi provides xplanition for the f st convergcnce beha-
vior o he self-orthogcnalizinc DFi donain scheme studied by
Picchi ana P ati [64] In cti 7 6 wc stow ho stimat s
of step length in the cis f iix a tep SD can be obtained
using cigcnv lues of th k-Circulan completions This explains
the obscrvation mad by l nan nd Schartz [21] regarding

fa t conv rgence of DFT domain schem compa ed to sample domain
sc r e

1 REVIEW OF STEEPEST DESCENT LGORITHM

In this s ction w discuss briefly c rtain g neral f atures
of he wellknown Cau hy s steep st d scent lgorithm [89] Con-
sid r th qu drati obj ctiv f nction giv n by (2 8)

$$F(x) = x^T H^T H\underline{x} - 2x^T_1 {}^T r - {}^- r \qquad (7\ 1)$$

The SD constructs a sequenc of v ctors starting with x_0
chos n a bitraril/ given by

$$\underline{x}_{1+1} = \underline{x}_1 + \alpha_1 H^T(\underline{r} \quad H x_1) \qquad 1 > 0 \qquad (7\ 2)$$

whr α_1 is det rmined such that $F(x_{1+1})$ is ninimized Substitu-
ting (7 2) in (7 1) and dirf rentiating w r to α_1 it can be
shown that α_1 is given b/

$$\alpha_1 = \frac{{}^T_1 e^1}{e_1^T H^T H\underline{e}_1} \qquad \underline{e}_1 = H^T(r - H\underline{x}_1) \qquad (7\ 3)$$

Computation of α_1 at ev ry iteration is the most time con
sum ng part of SDA Ther fore in most of th practical applica-
tion α_1 1 fix d at a value α chosen such th t th optimum
speed of c nverg nce is achieved It can b shown [89] that the
iteration scheme (7 2) conv rges to th LS solution

$$x_{LS} = (H^T H)^{-1} H^T \underline{r}$$

I may be int rest ng to note th t the fixed step SDA
(α = σ or all 1) is essentially a generalized version of the
Jacobi iteration [74] for solving the linear e uation

$$I^T H x = H^T r$$

Th iteration scheme in this cas ta es he fo r

$$x_{i+1} = (I_m - \alpha H^T H) x_i + \alpha H^T r \qquad (7\ 4)$$

This iteration scheme can be shown to converge f
$0 < \alpha < 2/\lambda_{max}$ Th o 1 um value o α is given by

$$\alpha = \frac{2}{\lambda_{max} + \lambda_{min}} \qquad (7\ 5)$$

where λ_{max} and λ_{min} are the maximum and minimum eigenvalues
of $H^T H$ respectively [90] (see page 24)

7 2 APPLICATION OF DFT TO STEEPEST DESCE T ALGOPITHM

√ now consider the possibility of using DFT to reduce
computational load associated with SDA Consider e_i given by

$$e_i = H^T (r - H x_i)$$

Let H_1 be the Circul nt completion of H and x_{1e} be the extended
vector formed by padding x_1 with n-m zero Then w can write

$$e_1 = H_1^T (r \quad H_1 x_{1e}) \qquad (7\ 6)$$

where $\underline{e}_1 = \begin{bmatrix} e_1 \\ - \\ e_{11} \end{bmatrix}$ and $\underline{e}_{11} = H_{11}^T (r - H_1 x_{1e})$ $H_1 = [H \quad H_{11}]$

From (7 6) it is clear that if we compute the ri ht hand

s de of (7 6) $\underline{e}_1$ is obtained as he first n ntries of $\underline{e}_1$

Jow let $\underline{E}_1$ denot the DFT of e_1 In erm of the DFT of r $\underline{x}_e$

and h the first column of H_1 we can write (7 o) in th DFT

uo nain as

$$\Xi_1 = FH_1^T \Gamma \ (Fr - \Gamma H_1 F \Gamma y_1$$

$$= F_1 \ (\iota \ H_1 \quad _{1e})$$ (7 7)

Tı r fore by knowing H_1 P anc $_1$ the r h hand side of

(7 7) is easily computed and e_1 is then obt ined as

$$e_1 = PF^* \underline{E}$$ (7 8)

wher $P = [I_{m \mid} O_{mxr -m}]$

Now if $\underline{E}_{1e}$ denotes th DFT of $\underline{e}_{1e}$ where e_{1e} is an exten-
ded vector obtained by padding $_{-1}$ with (n-n) z ros α_1 given by

(7 3) can be written as

$$\alpha_1 = \underline{e}_1^T e_1 / \ \overline{E}_{1e}^T \ (F_1^* \ H_1 \ \underline{E}_{1e})$$ (7 9)

fh ıs by knowing the DFT of e_{1e} computation of α_1 becomes
relativel/ simple task as we already have the knowledge of $\underline{H}_1$

It should be noted that the value of ι_1 remains same both
ın the sample domain ano th DF1 dor ain Accordingly the

convergence rate remains unaltered by the application of DFT
We will com back to this point later

Further we wish to note that as k-DFT computation using
FFT algorithm requires more computation ompared to DFT k-
Circulant completions other than k = 1 are not as useful in
reducing computations per it ration compar d to Circulant
completion

7 3 DERIVATION OF THE DFT DOMAIN STEEPEST DESCENT ALGORITHM

In this section we derive SDA in the DFT domain from three
diff rent viewpoints The obj ct ve is to cl arly bring out
th role of DFT in SDA

Approach 1 Using the extended vector $\underline{x}_e$ and Circulant comple-
tion H_1 of H we can write (7 2) al ernatively as

$$\underline{x}_{(i+1)e} = x_{ie} + \alpha_i \; P \; H_1^T \; (\underline{r} - H_1 \underline{x}_{ie})$$

where
$$P = \begin{bmatrix} P \\ O_{n-nxn} \end{bmatrix} \tag{7 10}$$

Premultiplying the above equation by th DFT matrix we
obtain

$$\underline{X}_{(i+1)e} = \underline{X}_{ie} - \alpha_i FP \; H_1^T (r - H_1 \underline{x}_{ic})$$

where $X_{ie} - Fx_{ie}$ Since $F^t F = I_n$ we can modify the above
expression as

$$X_{(1+1)e} = \underline{X}_{1e} + \alpha_1 FP\ F^* - \underline{\mathsf{H}}_1^T F\ (F\underline{r}\ FH_1 F\ Fx_{1e}) \tag{7 11}$$

Now $FH_1 F^*$ and $^rH_1^T\Gamma^*$ are diagonal m trices with DFT of th first column of H_1 and its complex conju ate a diagonal elements respectively Th refore we can writ (7 11) s

$$X_{(1+1)e} = \not{X}_{1e} + \alpha_1 FP\ F\ \mathsf{F}_1^* \ (\underline{P} - \underline{H}_1\ \underline{Y}_{1e}) \tag{7 12}$$

Moreover we can write (7 12) also as

$$\not{\underline{A}}_{(1+1)e} = \underline{\ }_{1e} + \alpha_1\ \underline{\mathsf{L}}_1 \tag{7 13}$$

where

$$\underline{\ }_1 = FP\ \Gamma^*\ _1 \tag{7 14)}$$

$$\underline{\mathsf{h}}_1 = \underline{\mathsf{h}}_1\ (\underline{P}\ -\ \underline{H}_1\ \not{\underline{\ }}_{1e}) \tag{7 15}$$

α_1 can oe expr ssed as

$$\alpha_1 = \underline{\Gamma}_{1e}^T\ \underline{\mathsf{E}}_{1e}/\ \underline{\mathsf{E}}_{1e}^T (H_1^*\ \underline{H}_1\ \underline{E}_{1e}) \tag{7 15}$$

The pair (7 13) and (7 1J) repre ents D T domain 1 ration schem s corresponding to (7 2' anc (7 4) respectively

Approach 2 In Approach 1 v derived DFT domain SDA starting from sample domain SDA In the pre ent approach we formulate the decoivolution p oblen as

$$\min_{\underline{z} \in R^n} \quad (r - H_1 z)^T \ (\underline{r} - Hz) \qquad (7\ 16)$$

subject to $Cz = 0$

where $\quad C = [0_{n-n \times m} \quad I_{n-m}] \qquad (7\ 17)$

The minimization problem given by (7 16) involves symmetric Circulant positive semidefinite matrix in general   It may be noted that eventhough $H^T H$ is positive definite $H_1^T H_1$ need not always be so   A simpl illustrativ xample is giv n below

Consider

$$H - \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} \quad \text{clearly} \quad H^T H = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \text{ is positive definite}$$

wh reas the Circulant completion of H

$$H_1 = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \quad \text{gives} \quad H_1^T H_1 = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & 1 & 2 \end{bmatrix}$$

which has 0 as an eigenvalue   Consequently $H_1^T H_1$ is positive semidefinite

It may be furth r noted that whereas mini ization of $F(\underline{x})$ given by (7 1) is an unconstrained proble (7 16) defines a constrained minimiza ion p oblem   Such problems can be solved

using Rosen   approach of projecting the gradi nt at every
iteration on to the null space of C [89] [91]   L t G b a
natrix such that CG = $O_{n-m \; n}$ From (7 17) 1 is easy to see
tl at Γ can have the following form

$$G = \begin{bmatrix} I_m \\ -- \\ O_{n-m \; m} \end{bmatrix}$$

Following the approach given in [89] (see pages 260 266)
w can write down the iteration scheme for (7 16) as

$$z_{1+1} = z_1 + \alpha_1 \Gamma G^T H_1^T (r - H_1 z_1) \tag{7 18}$$

wher $\alpha_1 = \min\limits_{\alpha} F(z_{1+1})$ but $GG^T = P$ We can now use DFT
and write (7 18) as done in th cas of Approach 1 in the
DFT domain   Thus We obtain

$$z_{1 \; 1} - Z_1 + \alpha_1 FP \; \Gamma^* (H_1^t \; r - H_1^* \; H_1 \; Z_1) \tag{7 19}$$

Since Z_1 plays the same role as X_1 in ("13) we s o that
(7 13) and (7 18) are identical

Approach 3   W now consider the approach taken by Valzman and
Schwartz [21] and Picchi ana Prati [64] for deriving DFT domain
SDA   Using F and the Circ ilant compl tion of F we can rewrite
th minimization problem given by (7 16) as

$$\min_{y \in R^n} (Fr \quad FH_1 y)^{*T} \; FF^* (Fr \; FH_1 y)$$

writing $FH_1\underline{y} = \Gamma H_1 F^* F\underline{y}$ and noting that $FF = I_n$ we obtain

$$\min_{Y \in C^n} (\underline{P} \, F_1 \, \underline{Y})^{*T} (R - H_1 \, Y) \qquad (7\ 20)$$

where by our earlier convention $\Gamma = Fr$ $F_1 = F\underline{h}$ and $\underline{y} = Fy$ Now in terms of $Y \in C^n$ we can formulate the minimization problem given by (7 16) equivalently as

$$\min_{\underline{Y} \in C^n} (P - H_1 \, \underline{Y})^{*T} (\underline{R} - F_1 \, \underline{Y}) \qquad (7\ 21)$$

subject to 1) $F \, Y$ a   real vector

and   2) $CF \, \underline{Y} = 0$

where C is same as in (7 16)   Solution to the above problem can be obtained using Rosen s gradient projection method as described in [21]   Alternatively since (7 21) is derived using (7 16) which has (7 19) as the corresponding SDA in the DFT domain   it follows that (7 21) also will have (7 19) as its corresponding SDA in the DFT domain

At this stage we wish to note the following distinctive feature of the three approaches   Approach 1 deals with a unconstrained minimization problem   In Approach 2 we deal with a constrained minimization problem with one constraint   In the third approach we have to deal with two constraints   The form of SDA in the DFT domain   however   remains unchanged   Likewise

he convergence parameter α_1 also remains unchanged by he application of DFT   Thus we can conclude ha DFT and th associated Circulant compl tion help only in reducing the computational compl xity of th iterativ scheme   These aspects donot b come clear ir the SDA 1 d rived using Approach 3 alone because solution of the rinimization probl m given by (7 21) do s not sh w cxplicitly its relationship with SDA in th   ample lomain

7 3 1 Data R covery Schem based on SDA in th DFT Domain

A DR schere based on DFT domain iter tion sch m (7 13) is sch natically shown in "igure 7 1   In this scheme the staiting point s chosen as

$$\angle_{oe} = \underline{R}|\underline{H}_1$$

Note that in the absence of noise and when the Cir ulant completion of H_1 is nonsingular $\angle_{oe}$ repres rts he DFT of the actual lata vector padded with n–m zeros   Thcrefor in such situations the iteration scheme w ll convergc in one itciation On the otherhand when noise is piesent $\underline{X}_{oc}$ provides a good starting point or he iteration s hem   The p rforrance of this scheme will o sane as that of DR sch ne based on LS solution
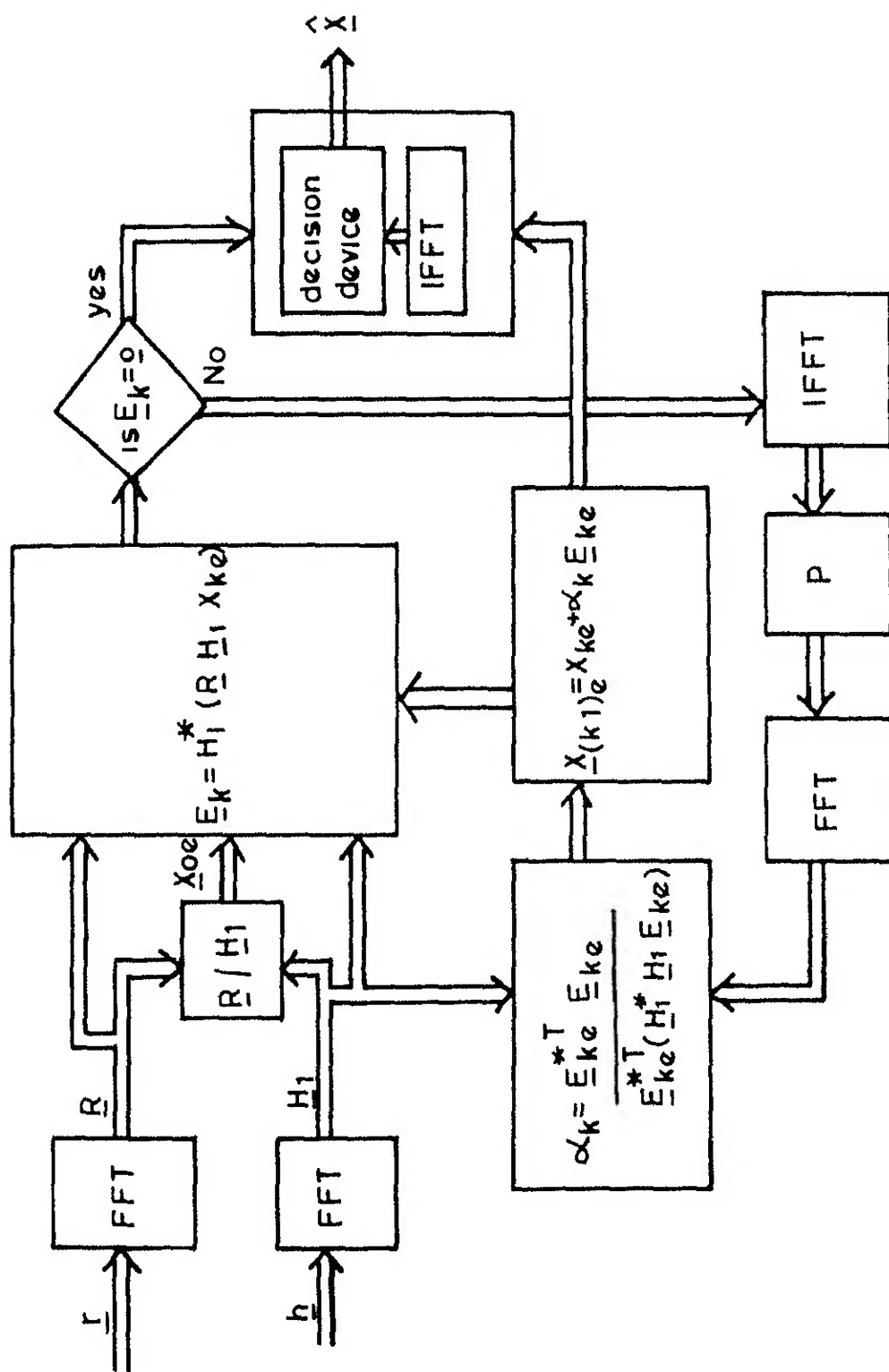
FIG 71 Data recovery scheme based on DFT domain steepest descent algorithm

7 4 RA1E OF CONV RGENCE OF SDA

ɔDA gener tes an infin1te sequence of $\underline{x}_1$ s whose l1m1 1s
the solut1on to the opt1m1zat1on problem Due to vere con-
stra1nts on proɔess1ng t1m 1n a commun1cat1on s1tuat1on 1arge
number of 1terat1on w1ll not be a des1rable feature Therefore
1 b come 1mportant to know the speed of convergence of SDA
In th1s connect1on we have the follow1ng tneorem [89]

Theorem 7 1   Suppose that $\tilde{x}$ 1s the solut1on to m1n1m1zat1on
of $F(\underline{x})$ g1ven by (7 1) L t c_n be the cond1t1on number of H^TH
That 1s $c_n = \lambda_{max}/\lambda_{min}$ Then (see [89] pag 146)

$$F(\underline{x}_{1+1}) - F(\tilde{x}) \le (\frac{c_n - 1}{c_n + 1})^2 \, [F(x_1) - F(\tilde{x})]$$

$$||\lambda_{1+1} - \overset{\sim}{\underline{x}}|| < c_n^{1/2} \, (\frac{c_n \, 1}{c_n + 1})^2 \, ||\lambda_1 \quad \overset{\sim}{\underline{x}}||$$

From Theorem 7 1 1t 1s clear that SDA has l1nc r rate of
conv rgence The rate of convergence depends on th cond1t1on
number of the channel autocorrelat1on matr1x   If $C_n = 1$ the
1terat1on scheme onverges 1n one 1terat1on   But as C_n 1n-
cr ases tne convergence r∂te decreases and hence large number
ɔf 1terat1ons w1ll be needed ɔefor mɔan1ngrul ɔst1mate of the
solut1on vector 1s obta1ned   Thus SD 1s not an eff1c1ɔnt
method when the cond1ᵗ1on numbɔr of ᴛ1ɔ c annel 1s 1arge One
methoɔ or ɔcɔ1erat1ng the convɔrgɔnce raᴛ of SDA 1s by

altering the eigenvalues of the channel autocorrelation matrix
so that condition number is near to unity   In th nɛx section
ʋ consider this aspect

7 5   FAST CONVERGING STɛEPᴸST DESCɛNT ALᴦORITHM

Consid r th it ration scheme given by (7 2)   Thɛ schᴦᴉ
conv ɪgɛs to the exact solution when

$$H^T(r - H\underline{x}_1) = \underline{0} \tag{7 22}$$

for some ɪ   Thɪ means hat in tᴄaᴄ or ℓ_1 we can substitute
αQ wher Q ɪs a nonsingular matrix and obtain another iterᴀ-
tion scheme

$$\underline{x}_{1+1} = \underline{x}_1 + \alpha Q H^1(r - H\underline{x}_1) \tag{7 23}$$

which will also give the sanᴉ solution as (7 2) provided α Q
and H satɪsry the n cessary conv rgent conditions   Jᴄ can
 how that the itᴄration schenᴉ (7 23) converg s if

$$||I_m - \alpha Q H^T H|| < 1$$

or $0 < \alpha < 2/\lambda_{max}$ where λ_{max} ɪs the maᴊɪmum eigenvaluᴄ of
$Q H^T H$

If $Q = (H^T h)^{-1}$ ɪt ɪs easy to see thᴀt the ɪtᴄ ation
scheme (7 23) convᴄrges to the s lution in one iteration if we
choose $\alpha = 2/(\lambda_{max} + \lambda_{min})$ where λ_{min} ɪs the minimum eigen-
value of $Q H^T H$ (ʋhᴄn $Q = (H^T H)^{-1}, \lambda_{min} = 1$)

It is clear f om the above discussion that different
choices of Q will lead to different iteration schemes   For
$Q = I_m$ we obtain fixed step SDA   Since computat on of the
best choice for Q which is $(H^T H)^{-1}$ itself is uneconomical
we must choose Q such that it is a good estimator of $(H^T H)^{-1}$
or the inverse of the channel autocorrelation matrix   Such
choices are difficult to ob ain in th sample domain

Now let us look at the DFT domain representation of
(7 23) given by

$$\underline{A}_{(1+1)e} = \underline{A}_{1e} + \alpha F Q_1 F^* E_{1e} \qquad (7\ 24)$$

where as before $\underline{Y}_{1e} = F \underline{x}_{1e}$ and $\underline{E}_{1e}$ is given by (7 14)
Q_1 is now an nxn matrix   From (7 24) we observe that if
$F Q_1 F^*$ is a diagonal matrix additional computation needed for
multiplying $Q_1 F^*$ and $\underline{E}_{1e}$ is just n multiplications   Therefore
in order to make $F Q_1 F$ diagonal Q_1 must be a circulant matrix
Since the best choice for Q is $(H^T H)^{-1}$ it seems natural to
choose for Q_1 $(H_1^T H_1)^{-1}$ whenever H_1 is nonsingular   This
feature has been exploited by Picchi and Prati [64] in their
fast converging self orthogonalizing DFT domain equalization
scheme in the context of channel equalization

So far in this se tion we discussed the application of
Circulant completion for obtaining good estimates of the
convergence parameter   However it may not be always possible

to obtain good estimate of the conv rgence parameter with
Circulant completion   In such cases k-Circulant completions
can be is d   For this purpose let $Q_k = (H_k^\lambda H_k)^{-1}$ We assum
that H_k 1 invertible   When $|k| = 1$ it can be shown that
Q_k is also a k Circulant   Therefore making use of k-DFT
(7 24) can b written using k Circulant completion of H a

$$\lambda_{(1+1)ke} = \underline{X}_{1ke} + \alpha F_k Q_k F_k^{-1} \underline{E}_{1l}$$

where $\quad \underline{X}_{1ce} \quad = F_k \underline{x}_{1k}$

and $\quad \underline{E}_{1ke} \quad = F_k P \ F_k^{-1} \underline{E}_{1k} \quad \underline{E}_{1k} = \underline{H}_k^* (\underline{P}_k - \underline{H}_k \underline{X}_{1ke})$

Since Q_k is a k-Circulant matrix $F_k Q_k F_k^{-1}$ will be a dia-
gonal matrix   It may b noted that use of k-Cir ulant comple-
tion r quires computation of k-DFT which can be computed u ing
FFT algorithm as described in Chapter 5   Th efficiency of the
iteration scheme depends on the value of k u co   A direct
 nalysis of th eigenvalue behaviour of Q_k and consequently its
effect on α is difficult as it depends upon the channel given
This aspect needs mor investigation

7 6   FIXED STEP STEPEST DESCE IT ALGORI THM

Th DFT domain version of fixed step SDA follows from our
discu sion in Section 7 3   In th s case we can write (7 13) as

$$\angle_{(1+1)e} = \underline{X}_{1e} + \alpha \underline{L}_{1e} \qquad (7\ 25)$$

The b st choice of α is given by (7 5)   As comput t on
of λ_{max} and λ_{min} is a difficult task   sti ites of th igen-
valu s of the channel autocorrelation matrix can be used   An
alternative method would be to choose the eigenvalues of $H_1^T H_1$
or $H_k^T H_k$ if th y are good estimate or the eigervalues of $H^T H$
One reason for such a choice is that they can b omputed
easily using DFT Walzman and Schwartz [1] have used eigen-
values of $H_1^T I_1$ in the iterative scheme However th y do not
seem to have noted that α obtained using $H_1^T h_1$ can also be used
in the sample domain

Let $\lambda_{max\ k}$ and $\lambda_{min\ k}$ be maximum and minimum igenvalues
of $H_k^T H_k$ and let

$$\alpha^{(k)} = \frac{2}{\lambda_{max\ k} + \lambda_{min\ k}} \qquad (7\ 26)$$

If $\alpha^{(k)}$ is c_ose to α th degradation in the convergence rate
o th iterative scheme will not be appreciable   Towards this
end we prove the following theorem

Theorem 7 2   Let λ_{max} and λ_{min} be the maximum and minimum
eigenvalues of $H^T H$ Then $\lambda_{max\ s} > \lambda_{max}$

$$\lambda_{min\ s} \leq \lambda_{min} \qquad (7\ 27)$$

where $\lambda_{max\ s}$ and $\lambda_{min\ s}$ represent maximum and minimum eigen-
values of $H_s^T H_s$ where H_s is a nonsingular square completion of H

Proof   We first prove that $||H^T H|| \leq ||H_s^T H_s||$

where $|| \ ||$ is the spectral norm of a matrix   Let $P = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}$

be an $n \times n$ matrix   Using P we can write

$$||H^T H|| = ||P H_s^T H_s P||$$

for $||P H_s^T H_s P|| = \left|\left| \begin{bmatrix} H^T H & 0 \\ 0 & 0 \end{bmatrix} \right|\right| = ||H^T H||$

Therefore $||H^T H|| < ||P||^2 ||H_s^T H_s||$

A P is a projction operator it has eigenvalue ither 0 or 1

Therefore $||H^T H|| < ||H_s^T H_s||$

If $\lambda_{max \ s}$ and λ_{max} are the maximum eigenvalues of $H_s^T H_s$ and $H^T H$ respectively we obtain

$$\lambda_{max} \leq \lambda_{max \ s}$$

That prov s the first inequality in (7 27)

W row show that $||(H^T H)^{-1}|| \leq ||(H_s^T h_s)^{-1}||$

we have $H_s^T H_s = \begin{bmatrix} H^T H_c & H^T H_c \\ H^T H_c & H^T H_c \end{bmatrix}$

Note that $H_s = [H \ H_c]$ Consider the matrix G

$$G = \begin{bmatrix} I & 0 \\ -(H_c^T H)(H^T H)^{-1} & I \end{bmatrix}$$

Then $||G^{-1}|| = 1$ Therefore

$$||(H_s^T H_s)^{-1}|| = ||(H_s^T H_s)^{-1}||\ ||G^{-1}||$$

Using the result $||AB|| \leq ||A||\ ||B||$ we obtain

$$||(H_s^T H_s)^{-1}|| > ||(H_s^T H_s)^{-1}\ G^{-1}||$$

$$= ||(GH_s^T H_s)^{-1}||$$

Now $GH_s^T H_s = \begin{bmatrix} H^T H & H^T H_c \\ 0 & H_c^T H_c - H_c^T h(H^T H)^{-1}\ H^T H_c \end{bmatrix}$

Therefore $||(GH_s^T H_s)^{-1}|| \geq ||(H^T H)^{-1}||$

Therefore $||(H^T H)^{-1}|| < ||(H_s^T H_s)^{-1}||$

Therefore $\lambda_{min\ s} \leq \lambda_{min}$

where $\lambda_{min\ s}$ and λ_{min} are the minimum eigenvalues of $H_s^T H_s$ and $H^T H$ respectively   **

In particular for k-Circulant completions we have

$$\lambda_{max\ k} \geq \lambda_{max}$$

and   (7 28)

$$\lambda_{min\ k} \leq \lambda_{min}$$

In a given situation for fixing α it may be desirable to try out different k-Circulant completions and choose the one which provides maximum α But this involves additional compu- tations   The problem of maximizing α over all possible

k-Circulant completions for a given channel does not appear to admit mathematically tractable solutions Thi as ect however needs further investigation

In summary we have discussed the role o DFT and k-Circulart completions in reducing the comput tional complexity and increasing th convergence rate of SDA We have showed how to obtain better estimates of the convergence parameter α using eigen values of k-Circulart compl tions of channel convolution matrix H As there is a one-to one correspond nce between sample domain and DFT domain iteration chemes esti-mates of α obtained via k-Circulant completion can be used in the sample domain itself to orov de faster convergence

SDA has only linear converg nce property and the rate of convergence deoends upon the condition number of the channel autocorrelation matrix In this connection elf orthogonali-zing scheme discussed in Section 7 5 could be useful

Considering the drawbacks of SDA despite its several advantages it appears that when there is severe constraint on proces ing time a better method will have to be chosen Th method so chosen must converge fast and it must have a better rate of convergence We wish to note that conjugate gradient algorithm (CGA) [89] meets thes d mands Howev r it is com putationally unattractive because each iteration takes more

processing tim compared to SDA Kobayashi [92] has consi-
dered CGA for adaptive channel equalization H cr CGA
docs not app ais to have received much att ntion in the ar a f
adaptive channel equalization du to its comput tional
complexity

CHAPTER 8

PERFORMANCE OF DATA RECOVERY SCHEMES

In this chapter we give a comparative account of performance of some of the data recovery schemes studied in previous chapters. Performance is specified in terms of probability of error (Pe). Since the probability of error depends upon block length, channel parameters, data recovery scheme employed, noise statistics and the rate of transmission over the channel, except in very special cases, in general, it is very difficult to obtain close form expressions for Pe. In this connection it may be recalled that it is possible to obtain close form expression for Pe for DR scheme considered in Chapter 5. On the other hand the schemes considered in Chapters 3 and 4 are non-linear in nature and hence are difficult to do. Under such situations simulation studies help in evaluating the performance of the DR scheme employed. Throughout this chapter we assume binary data transmission. However the analysis can also be extended to multilevel data.

This chapter is organized in the following manner. As a prelude to the study of the performance of various DR schemes in Section 8.1 we characterize good channels for BDT which introduce ISI and provide good performance subject to a given

criterion. There are two reasons for knowing such channels.
Firstly, they provide a benchmark for comparing the perfor
mance of other channels. Secondly one could use or filter
at the transmitter so as to make the overall response of the
channel and the prefilter to resemble closely a good channel.
Channel examples used in the simulation study are given in
Section 8 2. In Section 8 3 we discuss performance of BDT
systems using performance curves for a wide range of block
lengths and typical channel examples. In Section 8 4 we study
the effect of increasing ISI among the numbers of a block
which happens when the data rate over the channel is to be kept
the same as source data rate.

8 1 GOOD CHANNELS FOR BLOCK DATA TRANSMISSION

8 1 1   Maximum Minimum Distance Channels

Let S_x be the set of all possible m-1 length data vectors.
We know from our discussion in Section 2 4 that when r is
received in the presence of white Gaussian noise the best
estimate of the transmitted vector $\underline{x}$ is given by a vector
$\underline{x}_1 \epsilon S_x$ such that

$$|| \underline{r} - H\underline{x} || \leq ||r - H\underline{x}_j|| \quad \text{for all } \underline{x}_j \epsilon S_x \qquad (8\ 1)$$

Let S_y denote another set defined as

$$S_y = \{\underline{y}_j \mid y_j - Hx_j \quad \underline{x}_j \epsilon S_x\} \qquad (8\ 2)$$

wher v_j is the received vector corresponding to x_j when noise is absent. Let d_{ij} denote the distance between two distinct vectors y_i and y_j. Then

$$d_{ij} = ||y_i - y_j|| \quad i \neq j \qquad (8\ 3)$$

Due to the presence of noise the deconvolution rule (8 1) need not result in x_i which is the actual data vector transmitted. In order to minimize the occurrence of such situations it is desirable to have good separation between any pair of y_j s in terms of the distance metric given by (8 3). Let the minimum distance be d_{min}

$$d_{min} = \min_{y_i, y_j \in S_y} ||y_i - y_j|| \qquad (8\ 4)$$

Without loss of generality we will consider channel responses with unit length. A g+1 length vector $\underline{f}$ is called maximum minimum distance (MMD) channel sample response if

$$d_{min} = \min_{i \neq j} d_{ij} \quad 1 \leq i\ j \leq 2^m \qquad (8\ 5)$$

subject to $\underline{f}^T \underline{f} = 1$ and $-1 < f_i < 1$ $0 \leq i < g$ is maximum

The second constraint in the above formulation is necessary to avoid trivial solutions for example $f_i = 1$ and $f_j = 0$ for $j \neq i$ $0 \leq j$ g

d_{ij} in (8 3) can be written as

$$c_{ij}^2 = ||\underline{x}_i - \underline{y}_j||^2 = ||H\underline{x}_i \; Hx_j||^2 = (x_i \; x_j)^T H^T (x_i - \underline{x}_j)$$

Since convolution op ration is commutativ w can int r-chang the rol s of x_i and h and write

$$d_{ij}^2 = \underline{f}^T (\underline{x}_i - \underline{x}_j)^T (\underline{x}_i - \underline{x}_j) \; f \qquad (\quad \prime)$$

wh re $f_i = h_i \qquad 0 \leq i \leq g$

and $\underline{x}_i$ is the convolution matrix with $_{-i}$ in he first column L t

$$\underline{z}_{ij} = X_i - X_j \qquad (3\ 7)$$

it then follows that the elem nts of Z_{ij} are ernary variabl s (+2 0 -2) Moreover $\underline{z}_{ij}$ is also a onvolution matrix We can now write d_{ij}^2 alternatively as

$$d_{ij}^2 = \underline{f}^T \; Q_{ij} \; \underline{f} \qquad (8\ 8)$$

wh re Q_{ij} is a symmetric Toeplitz matrix Q_{ij} is given by
$$Q_{ij} = Z_{ij}^T \; Z_{ij}$$

In terms of (8 8) the optimization problem (8 5) o com s

$$\underset{\substack{\underline{f} \in R^{g+1}}}{\max} \quad \underset{\substack{i \neq j \\ 1 \leq i \leq 2^m}}{\min} \quad \underline{f}^T \; Q_{ij} \, f \qquad (8\ 9)$$

subject to 1) $\underline{f}^T \underline{f} = 1$ $\qquad (8\ 10)$

and $\qquad$ 2) $-1 < f_i < 1 \quad 0 \leq i \leq g$

The following lemma provides maximum value achievable if we solve (8 9) subject to (8 10)

Lemma 8 1   In the case of binary alphabet the maximum d_{min} achievable is at most 2

Proof   We make use of the inequality

$$\lambda_{min\ ij}\ \underline{f}^T\underline{f} \leq \underline{f}^T Q_{ij}\underline{f} \leq \underline{f}^T\underline{f}\ \lambda_{max\ ij} \tag{8 11}$$

where $\lambda_{min\ ij}$ and $\lambda_{max\ ij}$ are the minimum and maximum eigenvalues of Q_{ij}

Consider two data vectors which differ in only one position   In this case Q_{ij} is a diagonal matrix with all diagonal entries equal to 4   Therefore

$$\lambda_{min\ ij} = \lambda_{max\ ij} = 4 \tag{8 12}$$

Using the condition $\underline{f}^T\underline{f} = 1$ in (8 11) we get

$$\lambda_{min\ ij} \leq \underline{f}^T Q_{ij}\underline{f} \leq \lambda_{max\ ij} \tag{8 13}$$

From (8 12) and (8 13) it is clear that we cannot find an $\underline{f}$ for all $i \neq j$ such that

$$\underline{f}^T Q_{ij}f > 4 \quad \text{for}\quad \text{all } i \neq j \qquad\qquad **$$

Using Lemma 8 1 we prove the following theorem

Theorem 8 1   Determination of $\underline{f}$ which gives $d_{min} - 2$ is equivalent to solving

$$f^T Q_{1j} f \geq 4 \quad \text{for all } 1 \neq j \text{ and } \lambda_{min\ 1j} < 4 \qquad (\ 14)$$

$$\underline{f}^T f = 1$$

$$-1 < f_1 < 1 \qquad\qquad O \leq 1 \leq g$$

Proof   If $\underline{f}$ is chosen such that for all of those Q_{1j} with $\lambda_{min\ 1j} < \Delta$ we have

$$\underline{f}^T Q_{1j}\underline{f} \geq \qquad \text{clearly for the res of the } Q_{1j} \text{ s}$$

$$\underline{f}^T Q_{1j}\underline{f} > 4 \text{ holds from } (\ 13)$$

**

According to Theorem 8 1 the proc aure for obtaining MMD channels is to first determine all those Q_{1j} s for which $\lambda_{min\ 1j} < 4$ Then we have to solve the set of   dratic inequalities given by (8 14)   While good amount of literature is available on linear inequalit es [93] very little seems to have been achieved in understanding of systems of quadratic inequalities   We will not solve this problem directly in this thesis   Instead we will consider another characterization of good chann ls called maximum distance channls (MD-channels) le shall see later that MD-chann l haracterization is helpful in obtaining MMD channels

8 1 2 Maximum Distanc Channels

We no/ onsider those ch nnel which maximize the sum of
dis inct distances between all possible pairs of elements in
S_y That i if the pairs $y_i \, y_j$ and $y_p \, y_q$ give the same
distance $(d_{ij} = d_{pq})$ only one of the pairs will be considered
in the summation   In this case the problem may be formulated
as

$$\max_{f \, \varepsilon \, R^{g+1}} \sum_{Q_{ij} \; \text{distinct}} \underline{f}^T \, Q_{ij} \underline{f} \qquad (8\;15)$$

subject to (8 10)

put differently (8 15) can also be written as

$$\max_{f \, \varepsilon \, R^{g+1}} \underline{f}^T \, Q \, \underline{f} \qquad (8\;16)$$

subject to (8 10) where $Q = \sum_{Q_{ij} \; \text{distinct}} Q_{ij} \qquad (8\;17)$

In the presence of the second condition in (8 10) the
problem given by (8 16) is difficult to analy e   We therefore
solve the following simpler problem

$$\max_{\underline{f} \, \varepsilon \, R^{g+1}} \underline{f}^T \, Q \, f \qquad (8\;18)$$

subject to $f^T \underline{f} = 1$

A solution to problem (8 18) can be obtained by using
(8 11) from which it follows that the eigenvector corresponding

to the maximum eigenvalues is the desired solution to (8 18)
If this solution satisfies the second constraint in (8 10) then
a solution to (8 16) is obtained provided th minimum distance
achieved is 2 in the binary case   However h proc dure out-
lined above is computationally unattractiv for large block
sizes   Moreover it has not been no sible to obt in conditions
under which solution to (8 18) is also a solution to (8 16)
We feel that a more rigorous study may lead to interesting
result   In the absence of such r sults we have followed the
procedure outlined below for obtaining iD as well as MD
channels

8 1 3 Procedure for Obtaining MD and iD Channels

1   Given the block length m   and number of ISI samples g
         form the Q matrix given by (8 17)

2   Determine the normaliz d eigenvectors and the correspond
         ing eigenvalues of Q

3   The   eigenvector corresponding to the maximum eigen-
         value is a solution to (8 18)

4   Compute the minimum distance for each eigenvector satis-
         fying the second constraint in (8 10)

5   The eigenvector giving maximum minimum distance is a
         solution to (8 14) if the maximum minimum distance is
         2 in the binary case

It must be noted that we may not g t solutior to (3 14) as given by the last step in all the cases Towards this nd we conducted some numerical experiments to obtain channel r ponses using the above procedur for s ver 1 values of n and g Some of th s^ results are pr sented in Table 8 1 In the last column of the table we have given the minimum dis-tances achieved In all th^ cases 1 h ve obtained both MMD as well as MD channels Another int restin feature is that son of the channel responses are reminiscent of the p rtial response channels [94] For instance in the case of two ISI terms the channel responses (707 0 707) and (- 707 0 - 707) are duobinary channel responses The channels obtained for g = 4 5 6 also show similar features

The foregoing observation is interesting because partial response signalling schemes were or ginally suggested in order to achieve better spectral shaping of the transmitted signal As we do not have yet a formal definition of such schemes it is difficult to establish meaningful relation between spectral shaping and maximum distance separation This aspect needs further inv stigation At this point we wish to note that Folconor and Magoe [47] also have shown that optimum desired impulse of th channel and prefilter combination of leng h two under minimum mean square crit rion must b the duobinary impulse response

Table 8 1   Some examples of 1D and MMD Channels

m	g	Channel sampled response	α_{mn}	Remarks
6	3	707 0 0 0 707 -707 0 0 0 707	2 2	1D and 1D D
6	4	707 0 0 707 0 0 707 0 0 -707 0 0 0 0 -707 0 0 -707 0 0 -707 0 0 707	2 2 2 2	1D nd D 1D 1D and D 1D
6	5	-0 8165 0 0 0 40825 0 0 0 40825 0 57735 0 0 0 57735 0 0 0 57735 0 0 707 0 0 707 0 0	2 1 633 2	D D 1 D
6	6	-0 8165 0 0 0 408 0 0 0 408 0 0 0 577 0 0 0 577 0 0 0 577 0 0	2 1 τ 3	1 D D
7	7	-0 86603 0 0 0 2886 0 0 0 2886 0 0 0 2886 0 5 0 0 0 5 0 0 0 0 0 0 0 0	2 1 +14	1 1D 1D

MMD channels with $d_{min} = 2$ have another interesting property when used in conjunction with L-d convolution or Viterbi algorithm For such channels there will be essentially no loss in SNR when ML-deconvolution method is adopted This conclusion is based on the analysis given by Forney [38] H has shown that in the case of maximum likelihood sequence estimation the symbol error probability is bounded by

$$K_L \; Q(d_{min}/N_o) \; < \; P_e \; \leq \; K_U \; Q(d_{min}/N_o) \qquad (8\ 19)$$

where

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-y^2/2} \; dy \qquad (8\ 20)$$

and K_L and K_U are constants independent of noise variance and the channel parameters K_L is typically within an order of magnitude of K_U. We know that for one shot transmission (no ISI) the probability of error is given by

$$P_e(\text{one shot}) = Q(2||\underline{f}||/N_o) \qquad (8\ 21)$$

Thus if $d_{min} = 2||\underline{f}||$ the probability of error achieved in the presence of ISI using ML-deconvolution procedure will be approximately equal to P_e (one shot)

8 1 4 Computation of Minimum Distance of a Channel

A direct method of determining d_{min} is to compute the distance between all possible pairs of $\underline{y}$ s belonging to set S_y

given by (8 2)   This is a difficult job even for moderate
block lengths   We now show that ML-PDA d v l   d in Chapter 3
can be modified for computing d_{min}

From our earlier discussion in this chapter it is clear
that the problem of finding minimum distance can be formulated
as

$$\min_{\substack{x_i\ x_j \in S_x \\ i \neq j}} (x_i - x_j)^T H^T H (x_i - x_j)$$

Let $\underline{z}_{ij} = \underline{x}_i - x_j$

Elements of $\underline{z}_{ij}$ are ternary variables taking values from
the set $\{-2\ 0\ 2\}$ Let $\underline{z}$ denote a v tor whose elements are
ternary variables   In terms of $\underline{z}$ the above probl r can be
alternately formulated as

$$\min_{\underline{z}} \underline{z}^T H^T H\ z \tag{8 22}$$

$$\text{subject to } z^T z > 0 \tag{8 23}$$

Note that the constraint $\underline{z}^T z > 0$ is neces ary to eliminate
all zero solution for cas s corr sponding to $x_i = x_j$

The foregoing problem is a quadratic optimization problem
in the presence of a constraint   In order to find the non-
zero minimum value we adopt the procedure of Chapter 3 with the
following difference   In the fir t step when z_1 is optimized
for ev ry given value of $z_2\ z_3$ z_{g+1} choose only non zero

z_1 s Re t of the recursion stage are c rried out wi ! out any

constraint   This follows from the fact that s nce z_1 is con-

strained to take only nonzero values the constraint (8 23) is

satisfied and hence rest of the z_1 s can be hosen arbitrarily

We wish to note that in the context of CSDT Burkhart and

Barbosa [95] have sugges ed a nodified Viterbi alaorithm for

determining the minimum dis ance of a given hannel

8 2 CHANNELS USED IN THE STUDY

In order to compare the performance of diff rent data

recovery schemes it is necessary to consider channels which

show both merits as well as demerits of various DR schemes

The channel examoles considered are given in Table 8 2   In

iable 8 2 the first two channels A1 and A2 are maximum dis-

tortion channels considered oy Austin [13]   The next two

Table 8 2 Chann ls used in the simulation s id/

Channel sampled response	d_{min}
A1 $3^{-1/2}(1\ 1\ 1)$	1 633
A2 $5^{-1/2}(1\ 1\ 1\ 1\ 1)$	1 265
C1 $2^{-1/2}(-\ 235\ \ 667\ 1\ \ 667\ -\ 235)$	1 933
C2 $2^{-1/2}(\ 235\ \ 667\ 1\ \ 667\ \ 235)$	1 188
P1 $(\ 227\ \ 460\ \ 688\ \ 460\ \ 227)$	1 131
P2 $(\ 407\ \ 815\ \ 407)$	1 630

examples Cl and C2 taken from [4] Th 1 st two exa les
Pl and P2 are taken from [7]

Maximum distortion channels as the name it elf indicates
exhibit maximum realizable disto tion according to some distor-
tion measure Austin [13] determined such channels by choosing
the samples of the channel autocorrelation co ffici nts
a(-g)   a(O)   a(g) so as to maximize

$$\sum_{i=-g}^{g} |a(i)|$$

subject to 1   a(O) is unity

and   2 $A(w) = \sum_{i=-g}^{g} a(i) \exp -jiw)$

$$1+2 \sum_{i=1}^{g} a(i) \cos(iw) > 0$$

when w $\varepsilon(0\ \pi)$

The second condition implies hat the channel is realiza-
ble Austin has shown that the sampled auto orrelation fun-
ction having a triangular envelope corresponds to maximum
distortion channels in the sense defined above

8 3   SIMULATION STUDIES ON DATA R COVER\ SCHEMES

The following data recov ry schemes are considered

1   ML data recovery scheme for CSDT   } LC
2   ML data recovery scheme for BDT   MLB

3	Decision feedba k scheme for CSDT	DFC
4	Modified Austin s appioach for BDT	MAB
5	Fourier expansion method with decision reedback	FEB
6	LS data recov ry scheme for BDT	LSB
7	Circulant completion ba ed sche e for BDT	CCB
8	Skew Circulant completion bas d ch mc for BDT	SCB

Note that a three letter identirier is associ t d with each data recovery scheme These identifi rs have been used through-out in the text tables and figures We wish to note that LSB is based on Fourier expansion method without decision f edback In all the simulations 10000 data symbols taking binary (-1 1) values were used The curves plotted are obtained using single simulation

In Figures 8 1 - 8 9 w have shown the variation of th probability of error with SNR for the six channels given in Table 8 2 Performance curves in th case of MLC are obtained using ML-RDA given in Section 3 2 with the following modification D cision regarding a transmitted data is made after a fixed no of recursions say ID That is aft r the IDth stage we follow the path corresponding to the maximum value of the objective value achieved so for nd choose the data value orresponding to first stage After that we shift the ID stage results one stage back so that the second stage now becomes the
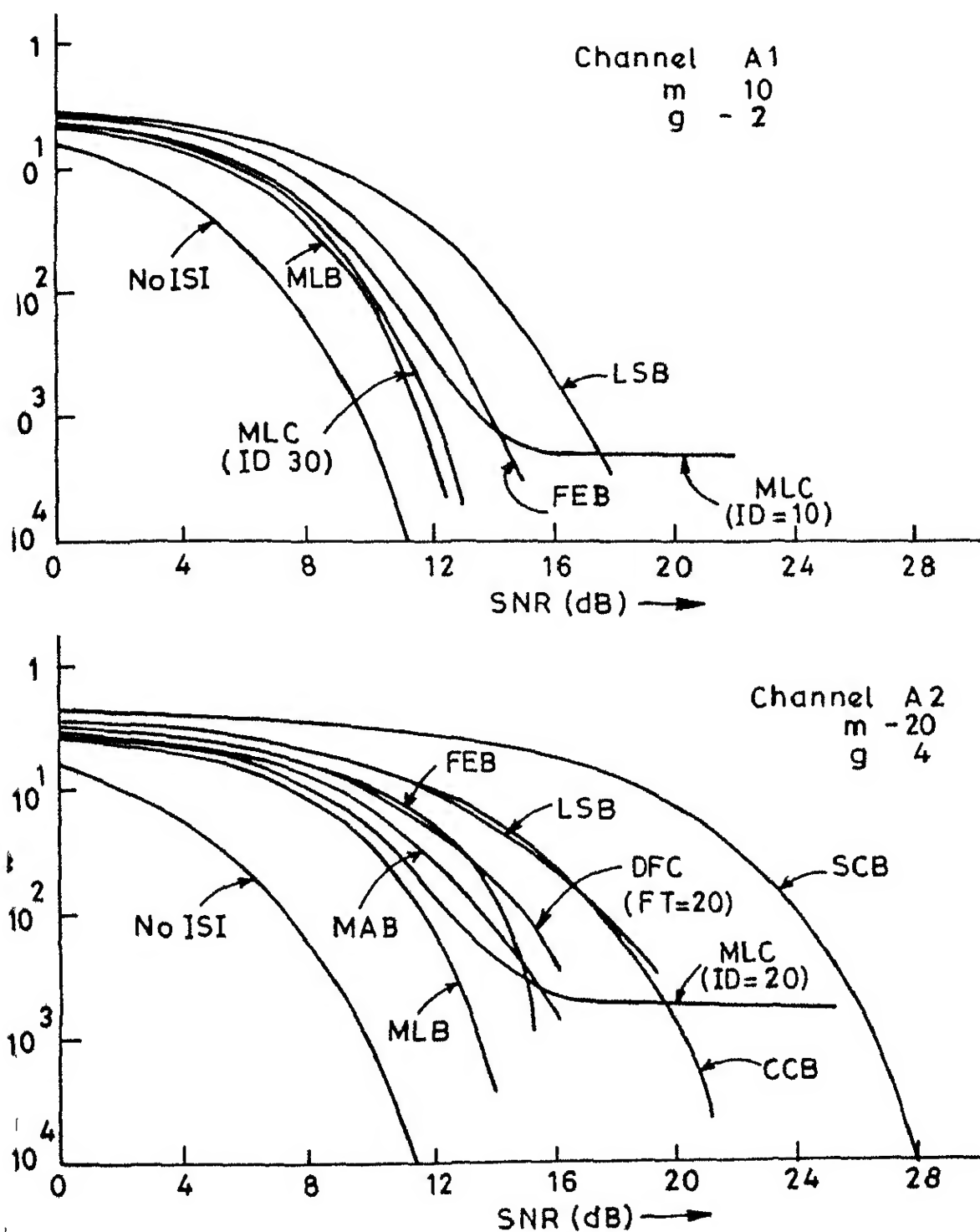
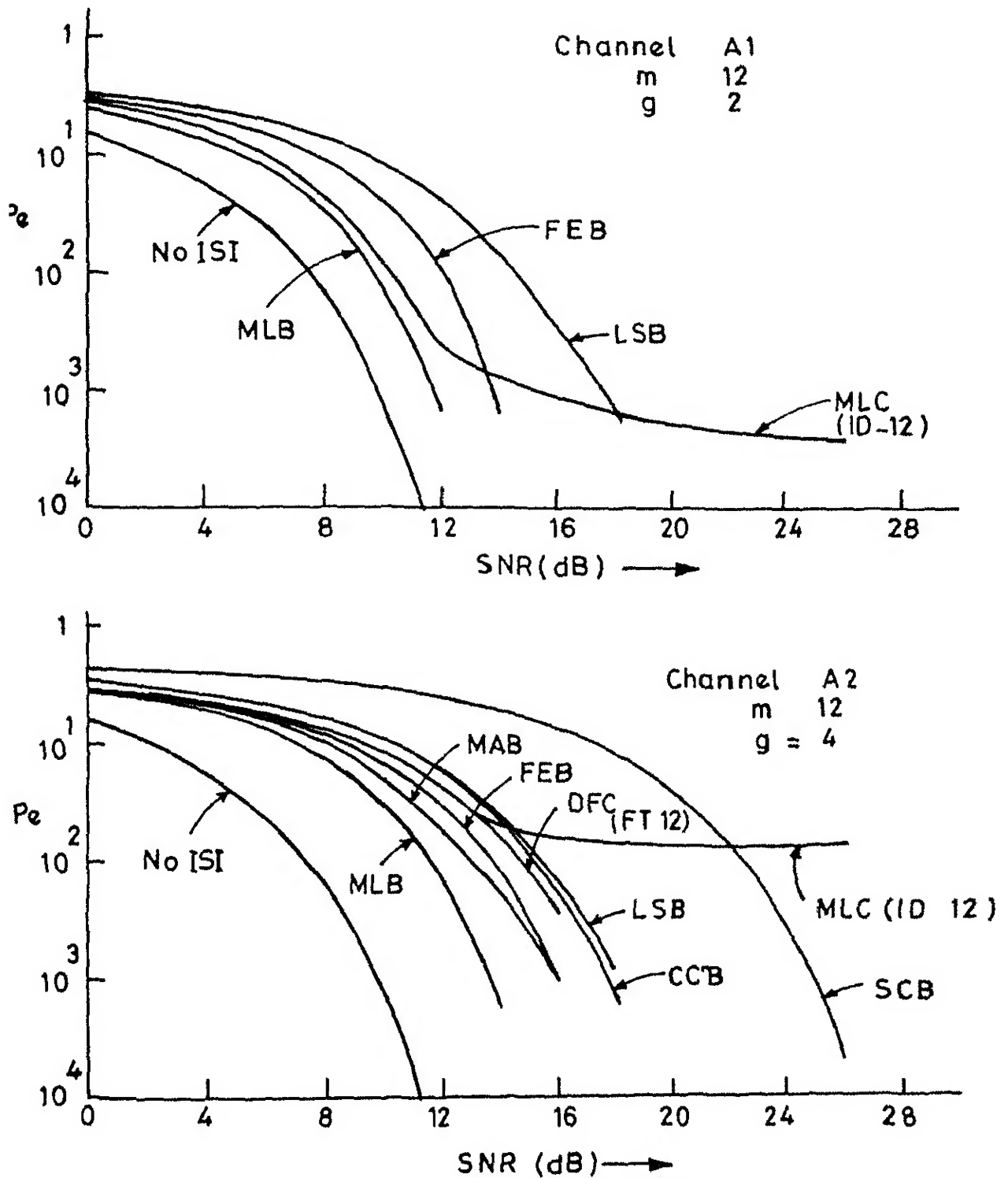FIG 8 1 Comparison of performance of various DR scheme
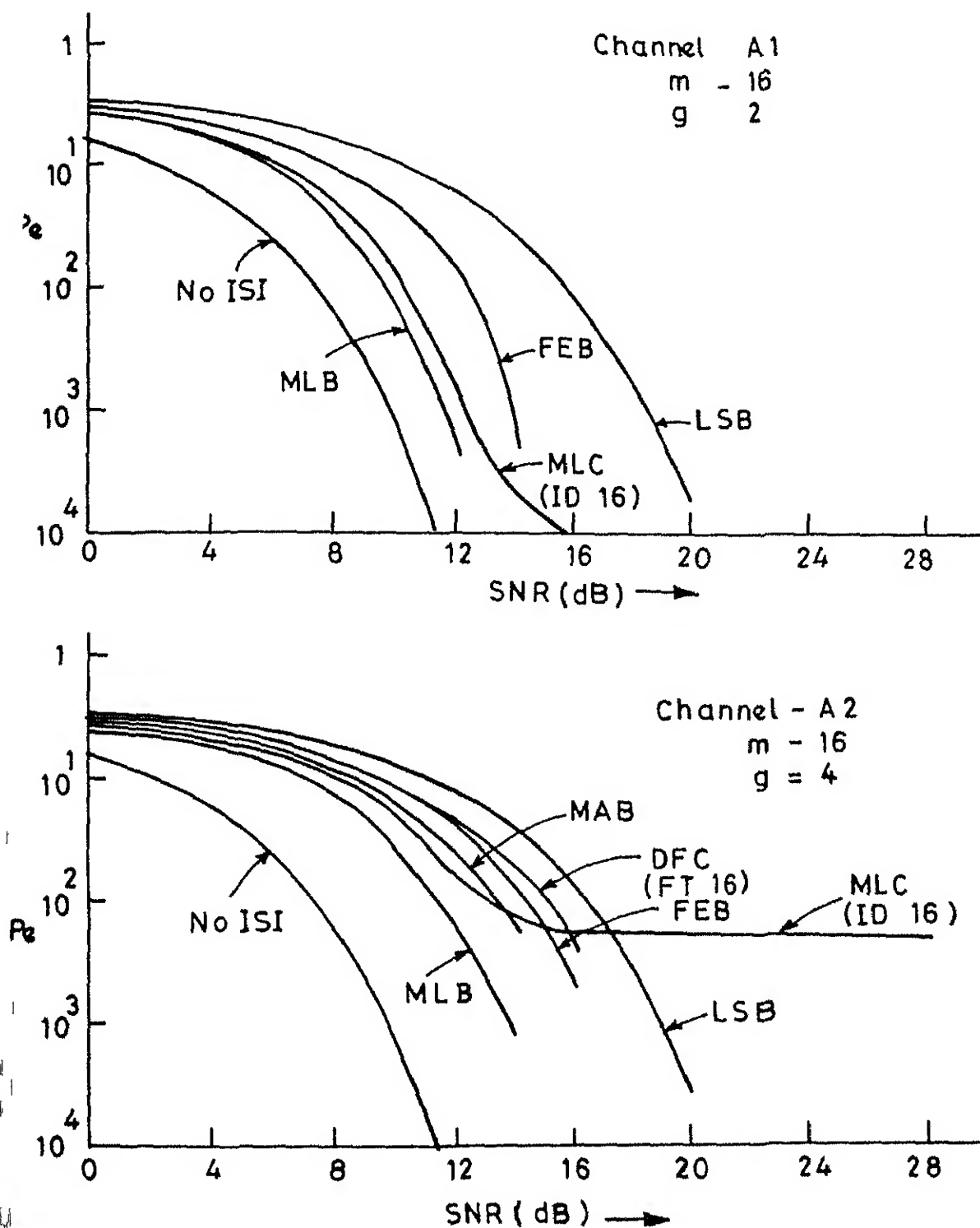
FIG 8 2 Comparison of performance of various DR schemes

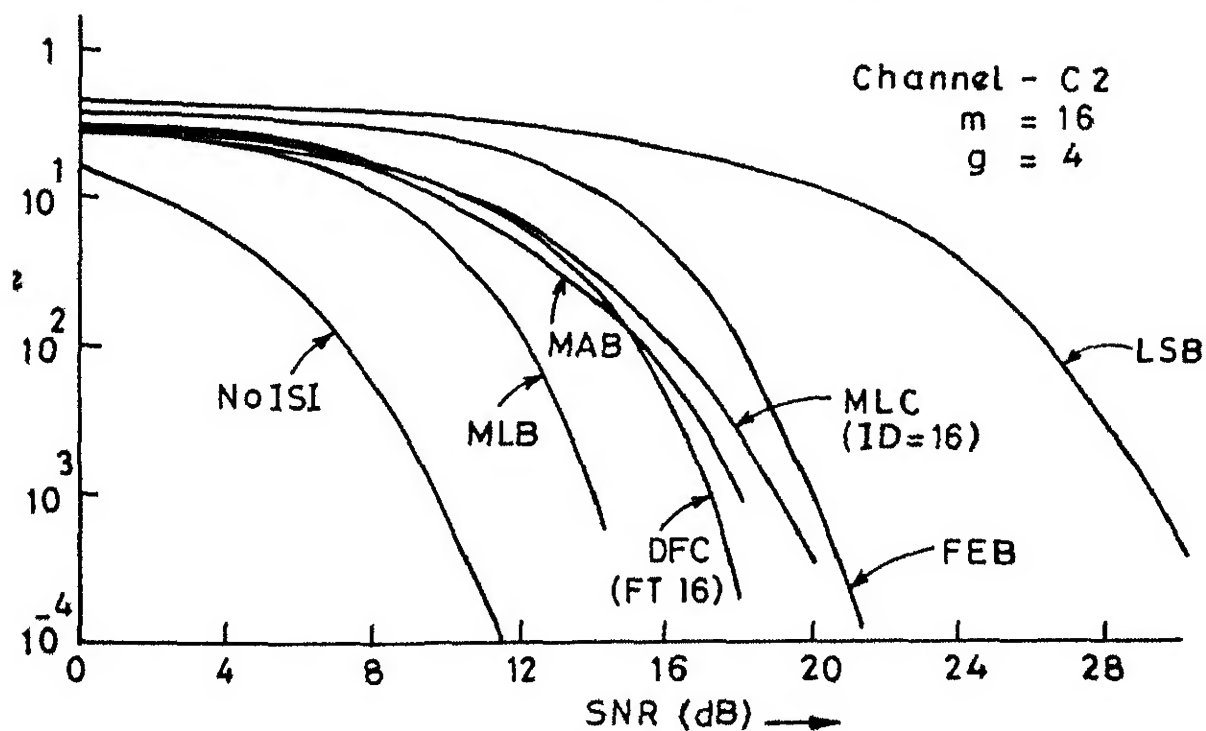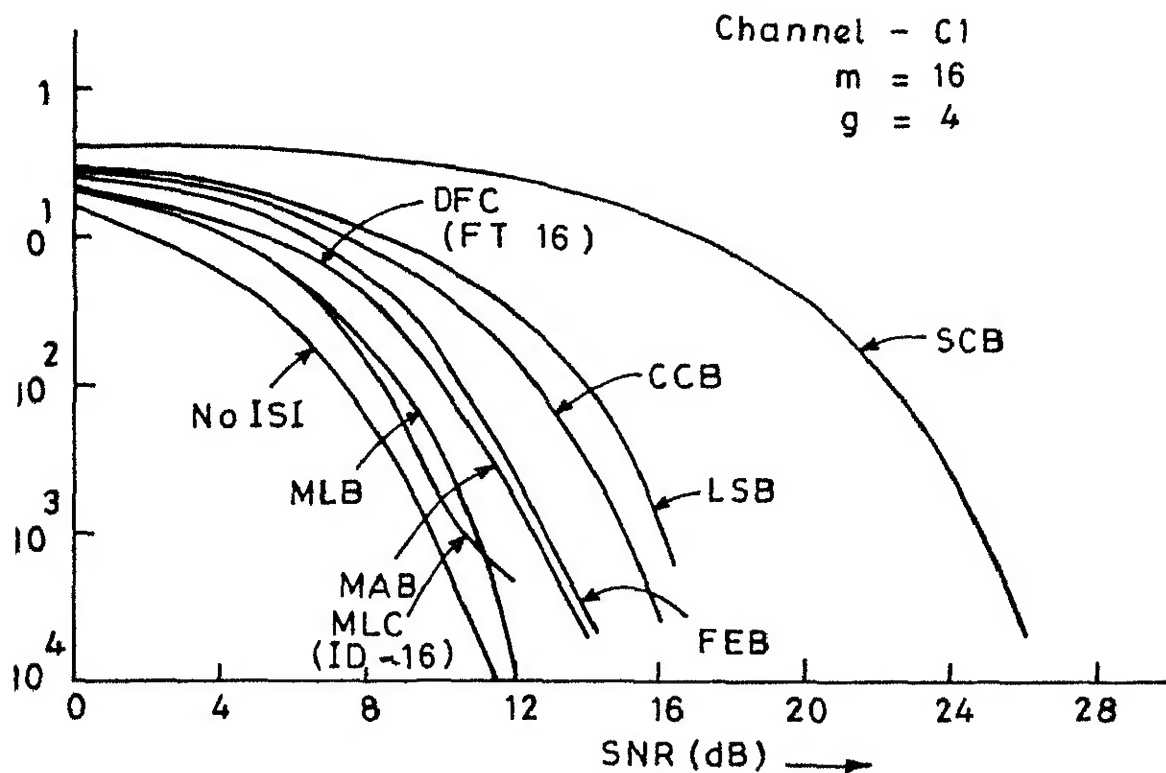FIG 8 3 Comparison of performance of various DR schemes

FIG 8 4 Comparison of performance of various DR schemes

The upper graph is labeled:

Channel C1
m – 12
g = 4

with curves labeled: No ISI, DFC(FT-12), LSB, MAB, SCB, MLB, MLC (ID 12), FEB, CCB. The vertical axis shows 10^{1}, 10^{2}, 10^{3}, 10^{-4} and the horizontal axis SNR (dB) from 0 to 28.

The lower graph is labeled:

Channel – C 2
m = 12
g = 4

with curves labeled: No ISI, MLB, MAB, FEB, MLC (ID=12), LSB, DFC (FT 12). The vertical axis is labeled P_e with 10^{1}, 10^{2}, 10^{3}, 10^{4} and the horizontal axis SNR (dB) from 0 to 28.

FIG 8 5 Comparison of performance of various DR schem

FIG 8 6 Comparison of performance of various DR scheme

FIG 8 7 Comparison of performance of various DR schemes

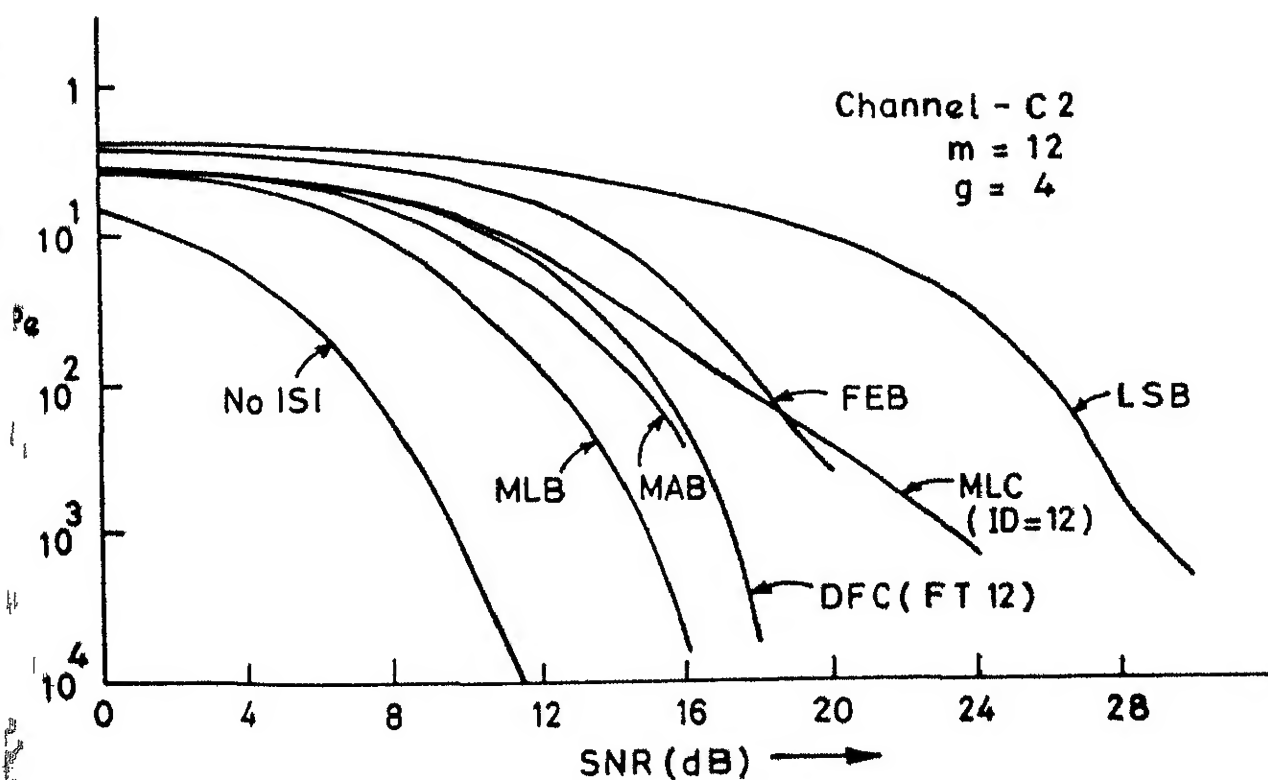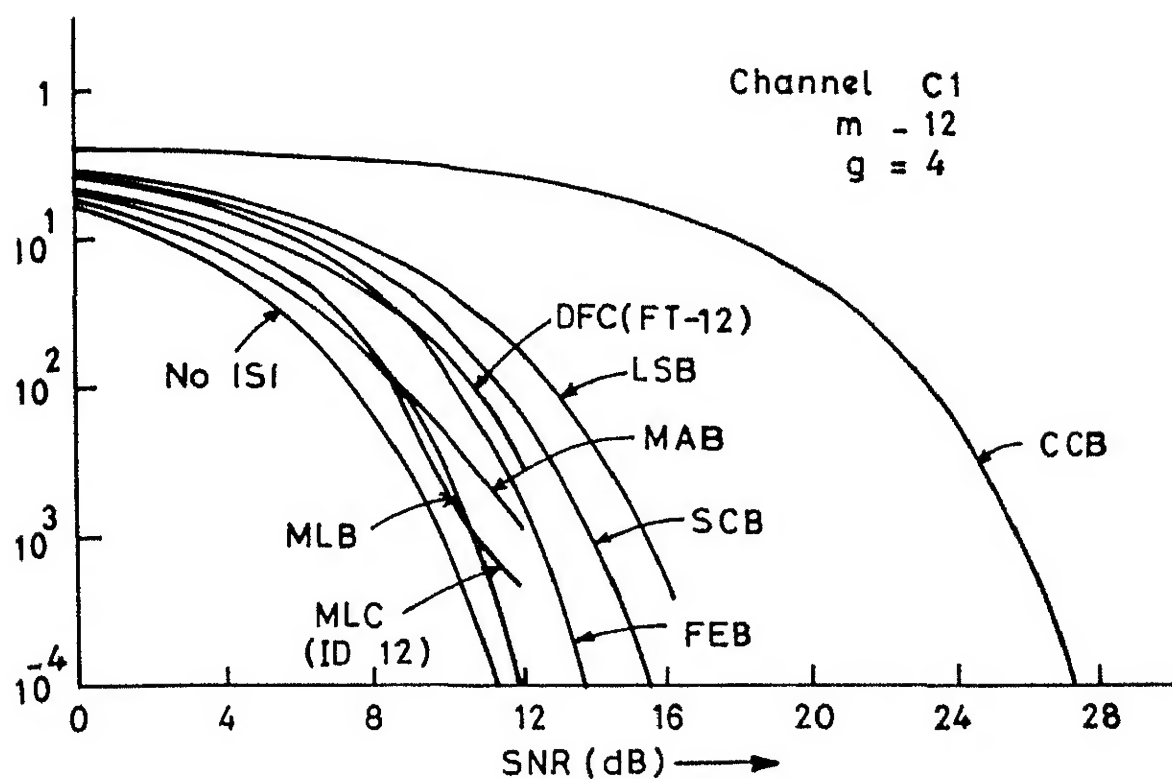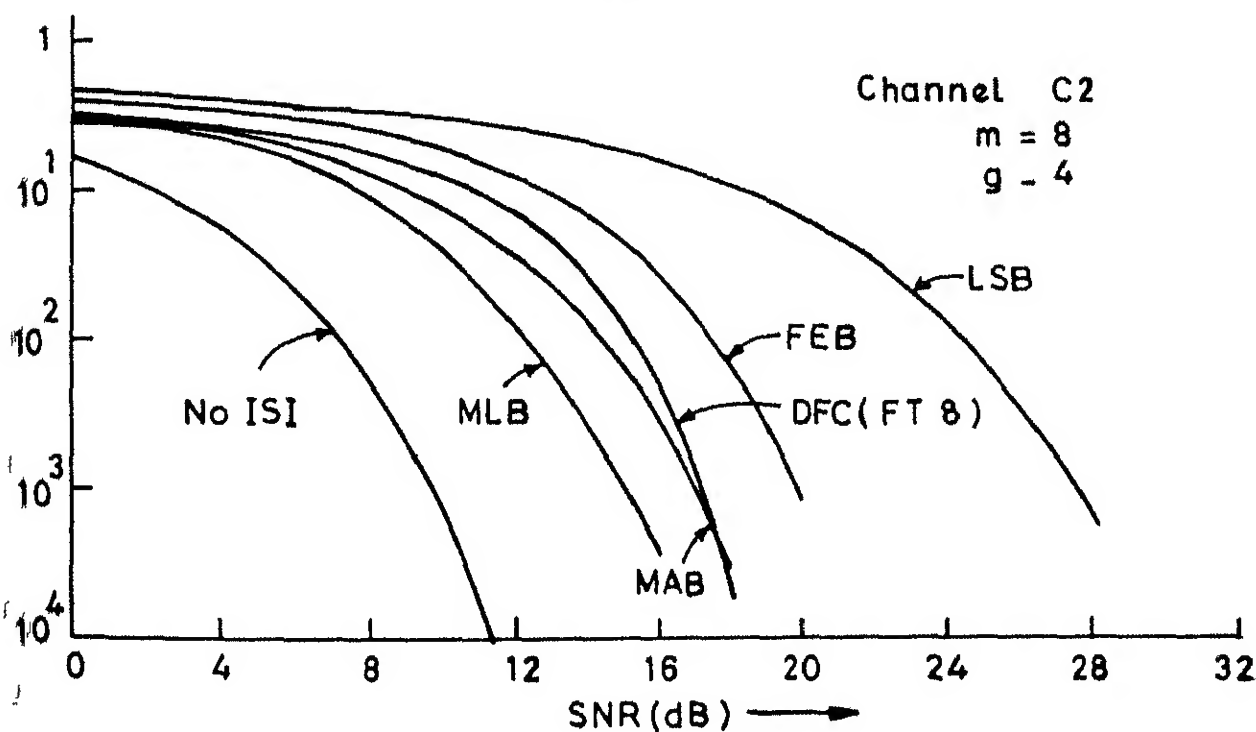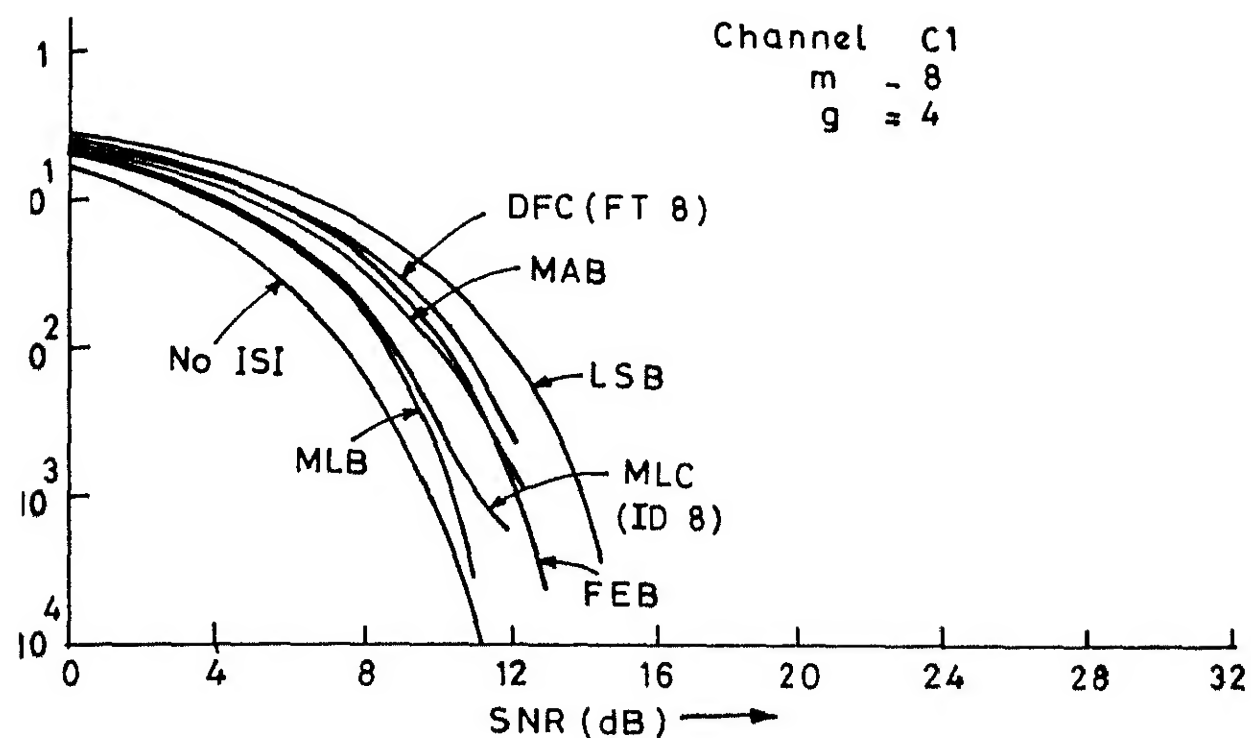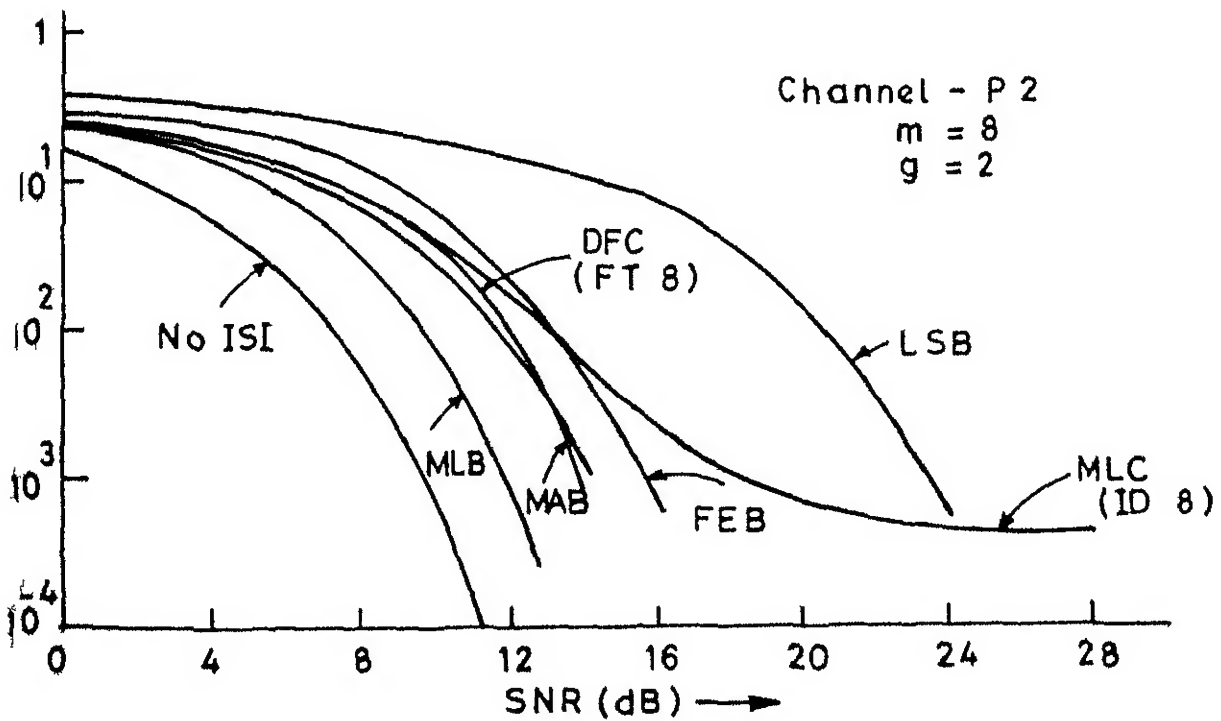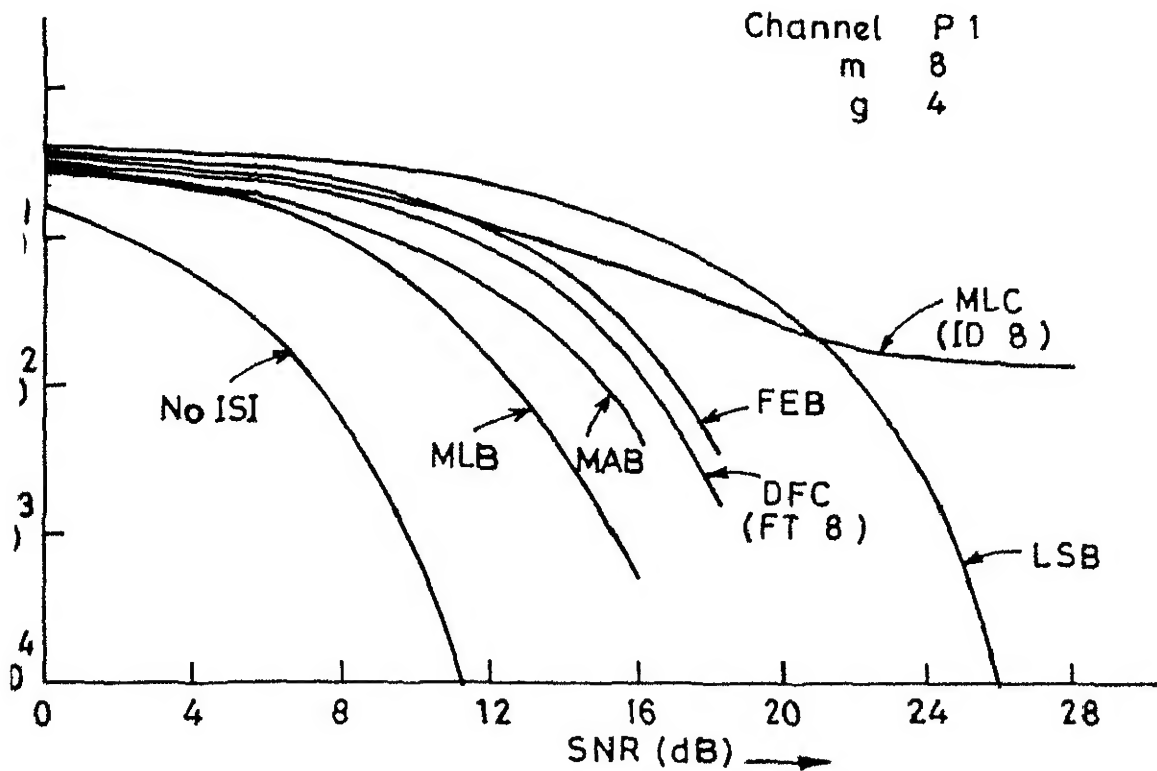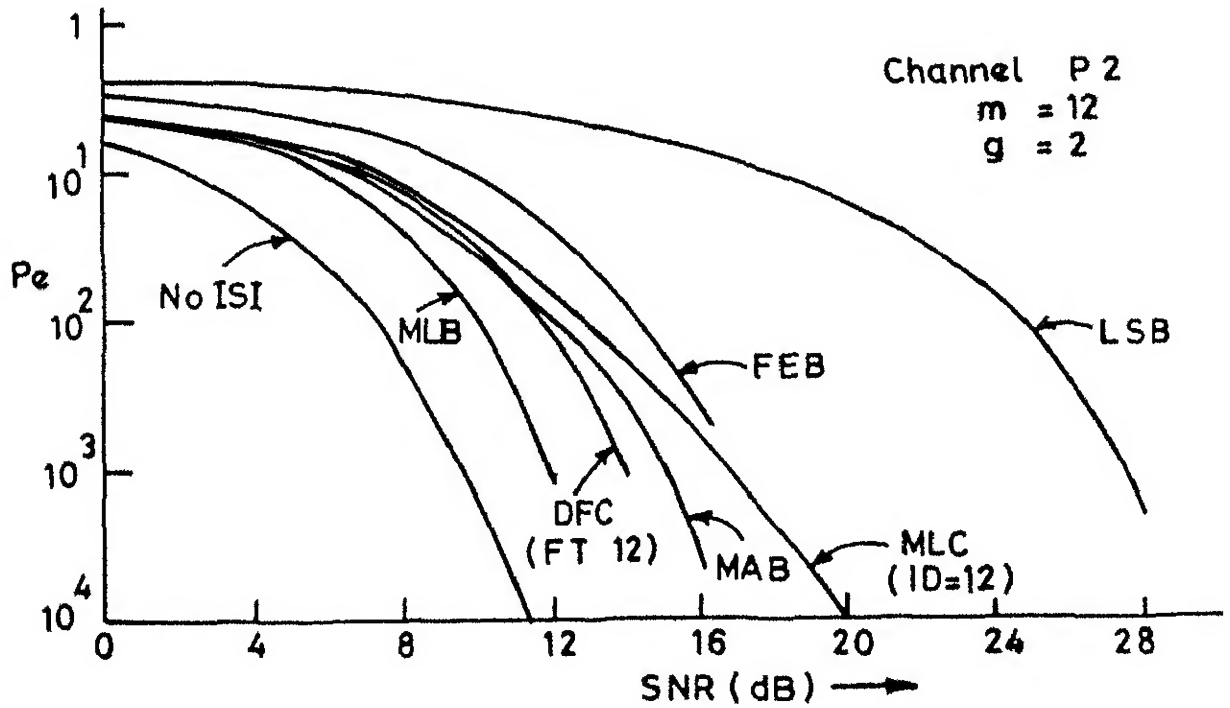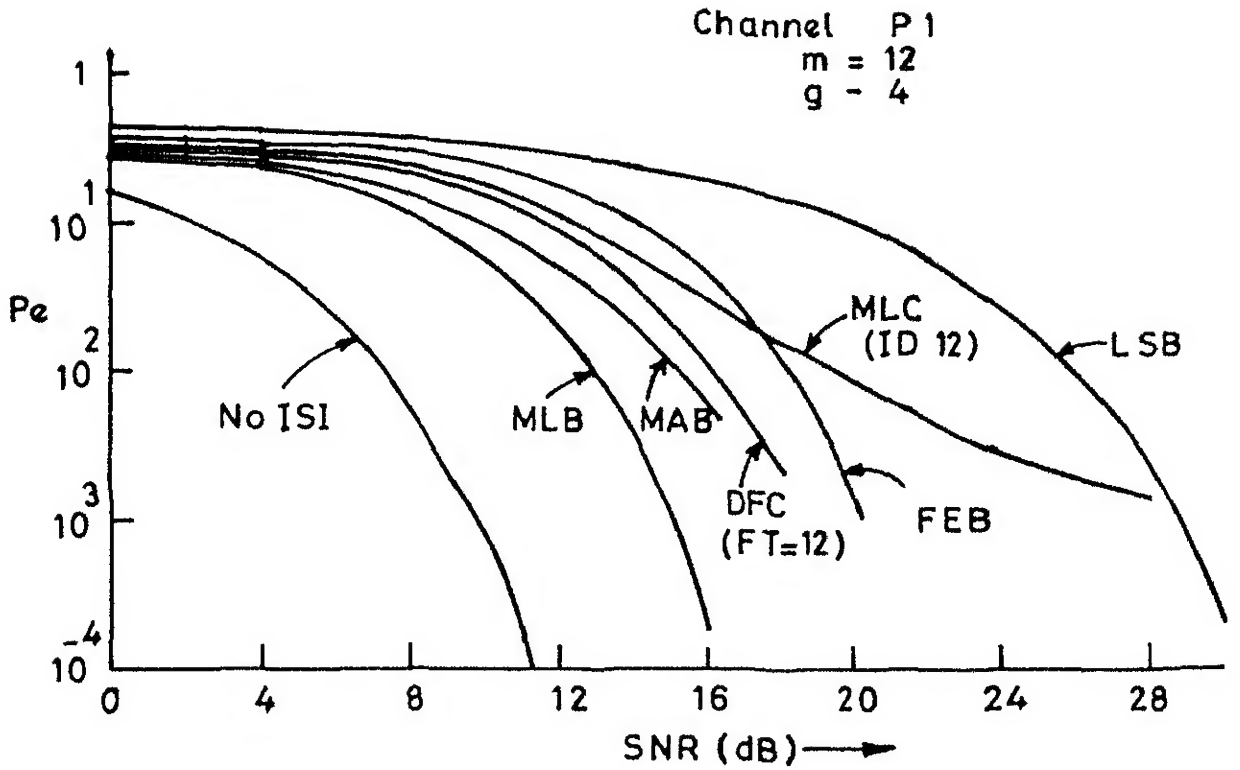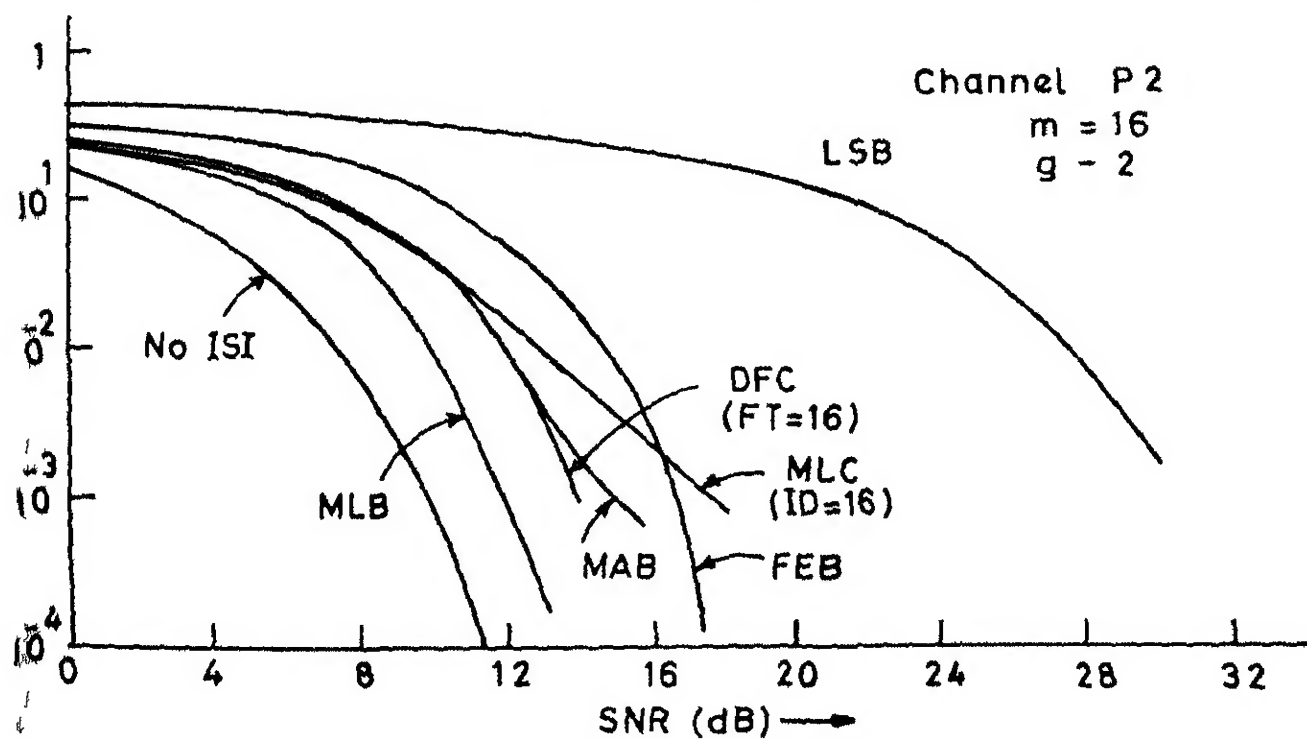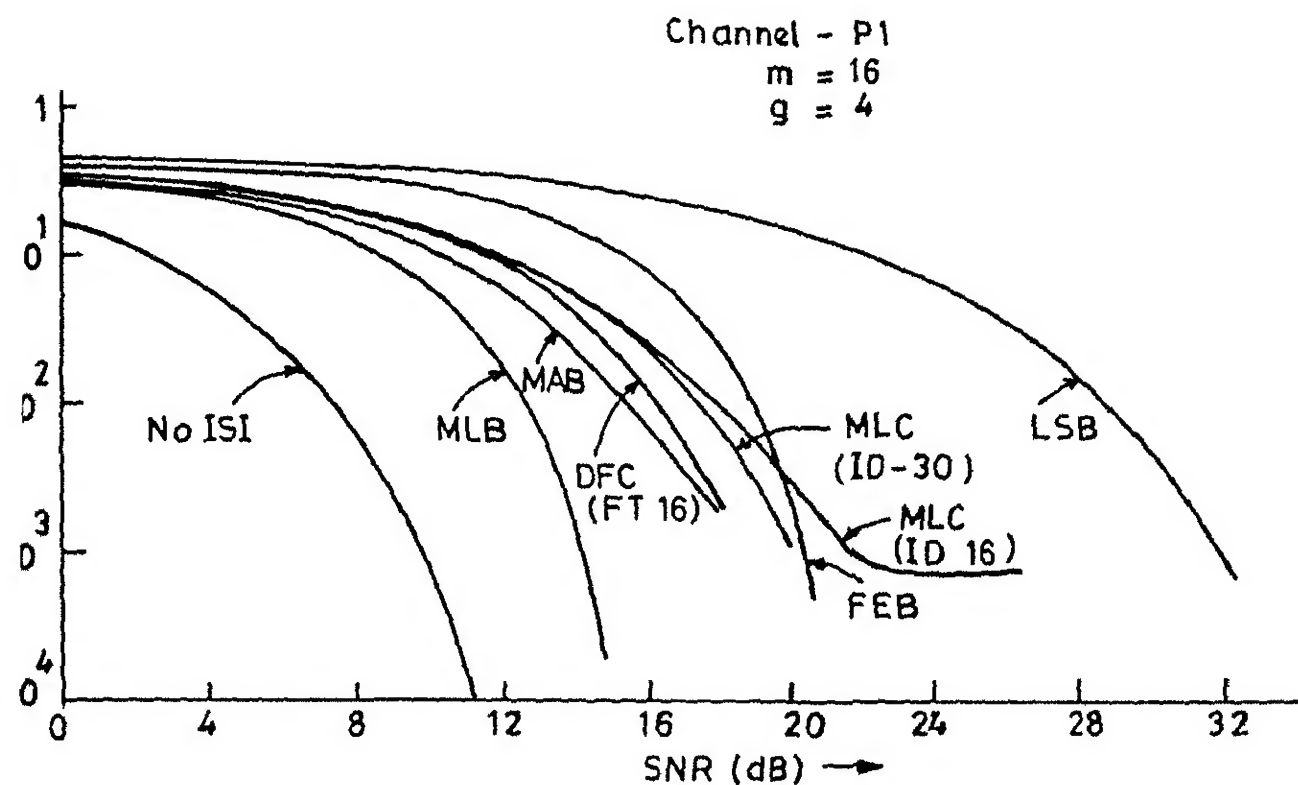FIG 8 8   Comparison of performance of various DR schemes

FIG. 8 9 Comparison of performance of various DR schemes

first stage   The procedure is continued till all the trans-
mitted data are decided   The value of ID i m it oied in the
figures   The decision feedback scheme for C DT (DFC) is based
on the equations given in [7]   The number of forward taps used
is den ted by ⁻T in the figures   For th   ake of co pari on error
probability curve for channels with no ISI is lso shown in
the figures

The Figures 8 1 to ⁻ 9 cover a sufficiently wide rang of
block lengths and channel responses   The figures indicate that
the performance of MLB is better than MLC when ID is equal to
the block length   In the course of the simulation studies we
have observed that for obtaining good performance at medium and
large SNR values ID must be large in the case of MLC   In
Figure 8 10 we have compared the performance of MLB and MLC on
th maximum distortion channel $\frac{1}{\sqrt{5}}$ (1 1 1 1 1) Figure 8 10 shows
that MLC performance saturates if ID is not la g

From the Figur s 8 1 - 8 9 we also observe that MAB performs
well compared to DFC when the block length is equal to the number
of forward taps   However it must be kept in mind that MAB
requir s more memory compar d to DFC   It may b recalled that
the computational complexity of both he schemes are more or
less same   FEB how ver does not p rform as well as MAB   Onc
reason is that FLB assumes no knowledge of the noise variance
One attractive feature of FEB is that it can handle illconditio-
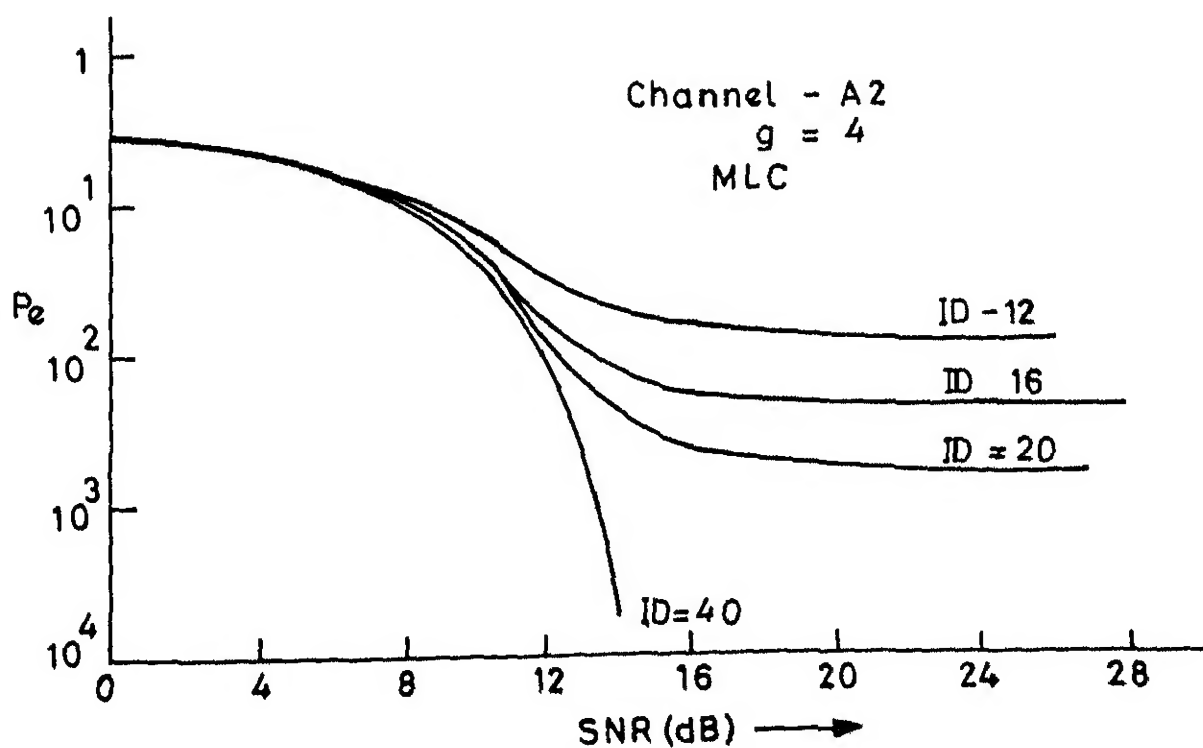ned channels more efficiently compared to the Trench algorithm

Channel – A 2
g = 4
MLB

m – 16
m = 12
m – 20

P_e

SNR(dB) ⟶

Channel – A 2
g = 4
MLC

ID – 12
ID 16
ID ≈ 20

ID = 40

P_e

SNR (dB) ⟶
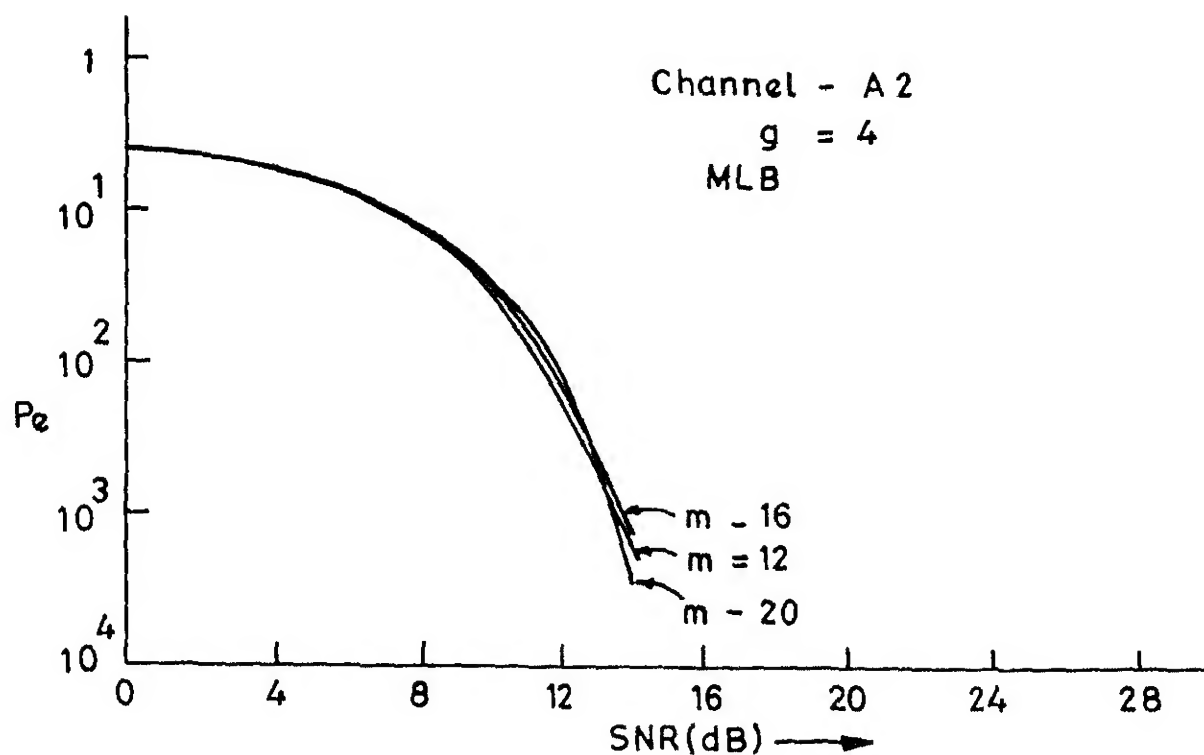
FIG 8 10 Comparison of MLB and MLC performace

based MAB in which illconditioned ch nnels can j /e r se to large values of the tap coefficien s

It is interesting to note that in the case of Cl (see Table 8 2) there is essentially no differ nc oet/een MLC and MLB for block length of 8 12 16 In fac CC performance is also very close to MLB One reason for t is perhaps is that d_{min} for this channel is 1 93 Whereas for the other channels it 1 much less than 2 (see Table 8 2) In order to check the validity of this claim in Figure 8 11 we have plotted the performance curves fo the example given in {33} For this channel d_{min} is 2 Performance curves indicate that the loss in performance due to CCB compared to MLB is not ignificant This observation however needs more inv stiga- tion

Figure 8 12 shows the effect of block length on the per- formance of CCB and SCB in the case of channels Cl and Pl Although the performance in the case of channel Cl (d_{min}=1 93) is better compared to that with Pl (d_{min} = 1 13) the perfor mance depends strongly on block l ngth

8 3 1 The Effect of Increased ISI Among the Block Elements

So far we have considered BDT in which the effective data rate over the channel is less than the source data rate due to g zeros following each block N now consider the situation in which the data values of a block are transmitt d at a higher
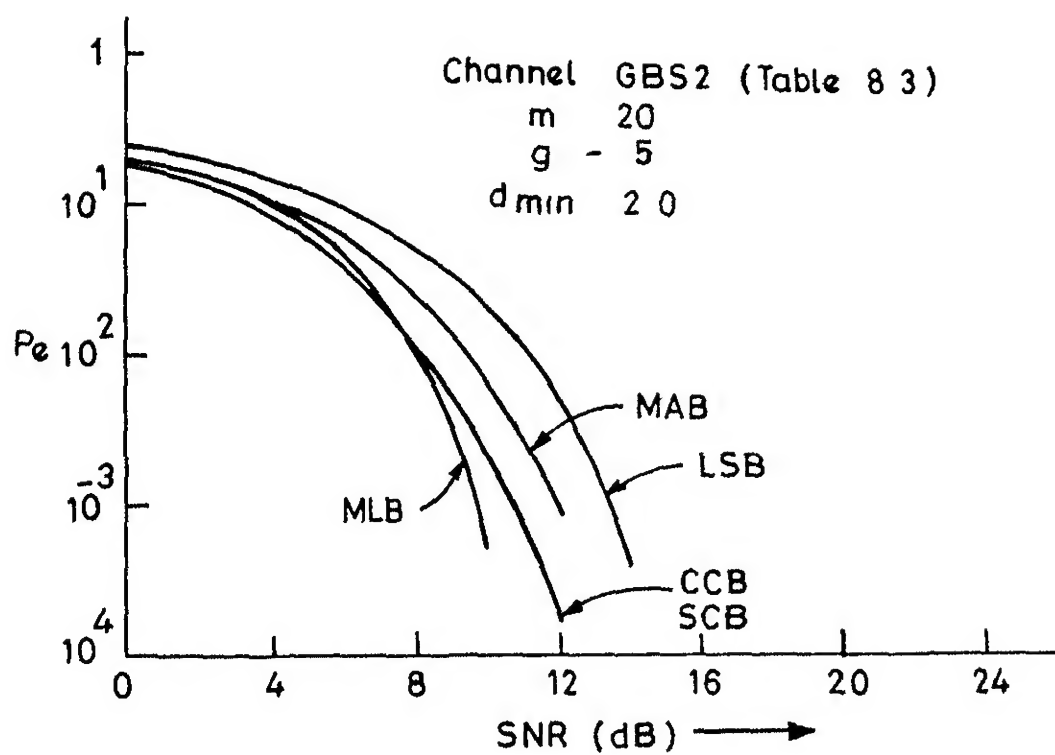
FIG 8 11   MMD Channel performance
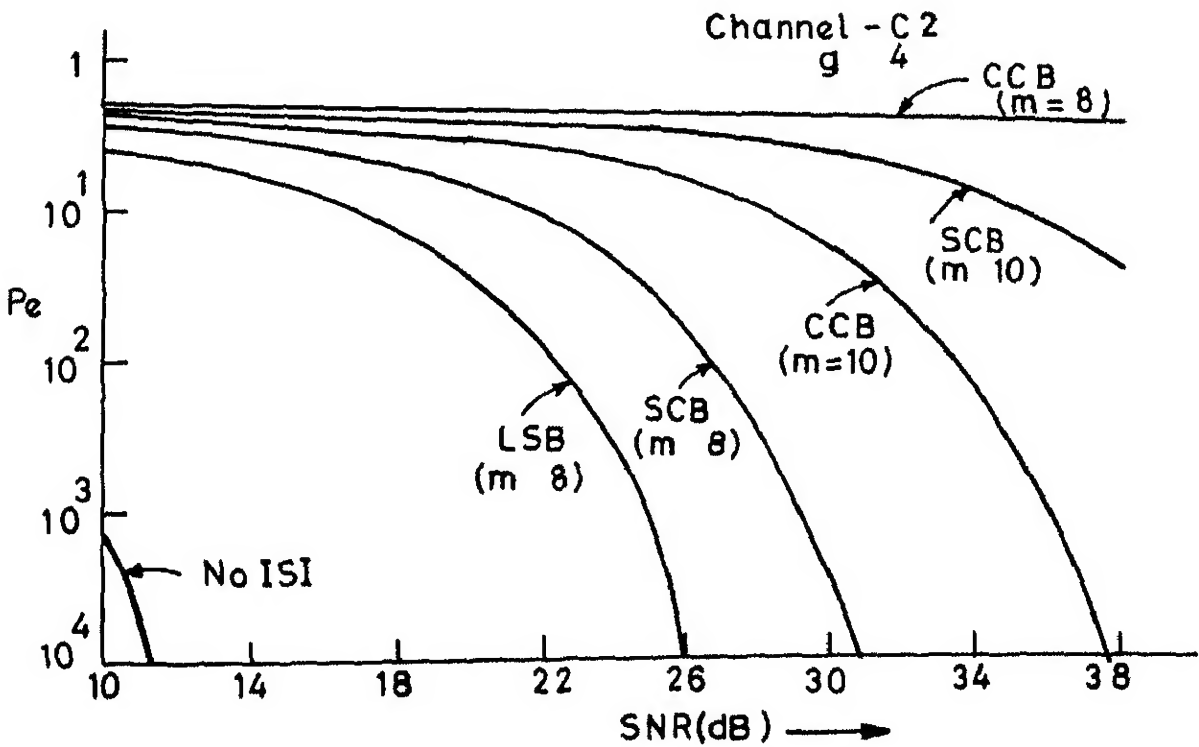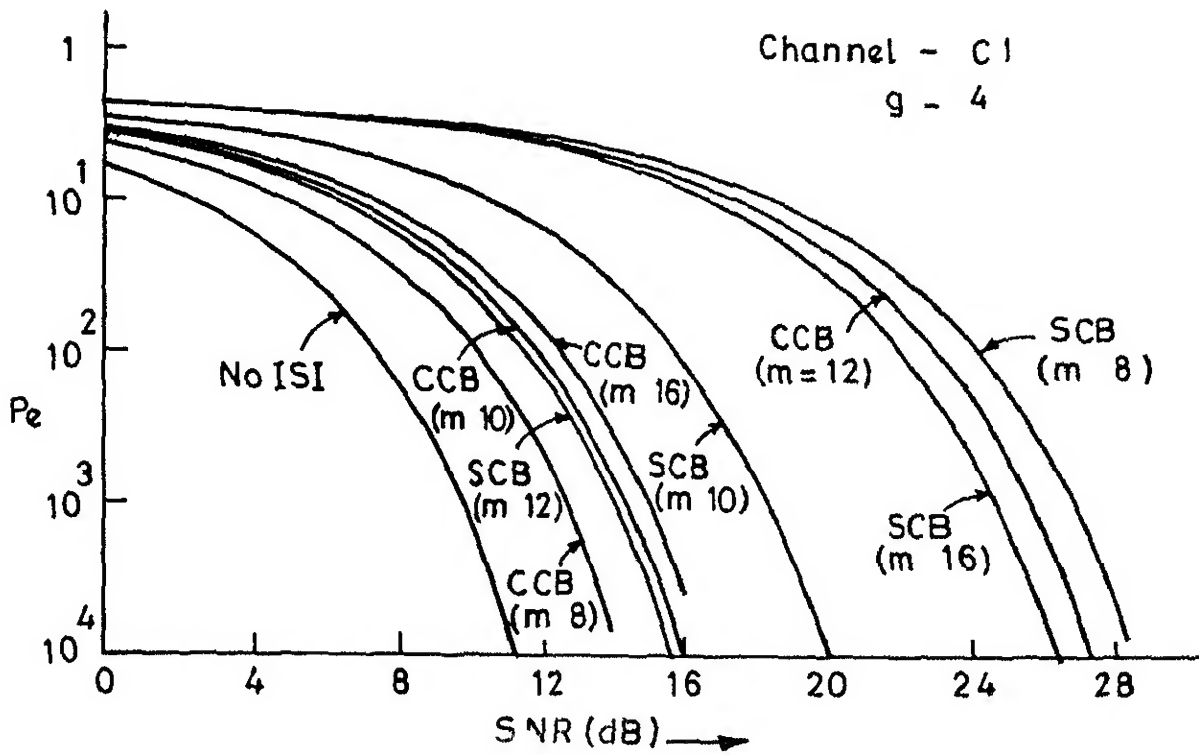
FIG 8 12 Effect of m on the performance of CCB and SCB

rate so that there is no reduction in th d a r t over the
channel However this results in increased IoI among the
block members The effect of incr asing the data rate on the
performance is the topic of this section

First we consider maximum distortion channels In
Figure 8 13 we have shown the effect of increasing ISI when
MLB is employed For the sake of comparison we have also
plotted MLC performance for ID equal to the block length Note
that the MLC curve for ID = 16 aturates aft 14 dB

Figure 8 14 shows the perforance curves in the ca e f
DFC and MAB In this case also we obs rve that there is
improvement in the performance of MAB compared to DFC

As another example we considered a telephone channel
given in [33] In Table 8 3 we have given the equivalent

Table 8 3 Channel sampled response coefficients

Channel	Symbol duration (in nano-seconds)	Channel par meters						d_{min}
GBS1	3	013	- 067	074	403	910		2 0
GBS2	2 5	01	044	- 094	031	061	821	2 0
GBS3	2	017	044	- 086	272	676	678	1 719
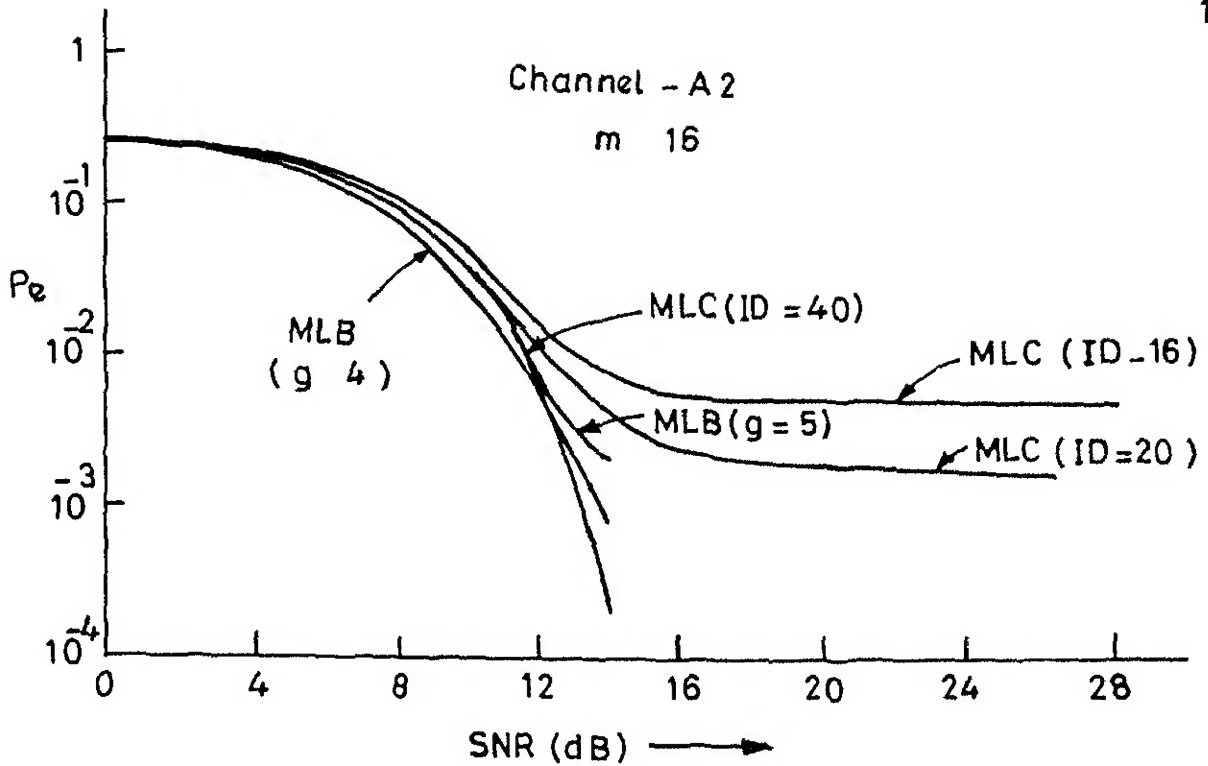
FIG 8 13 Effect of increased ISI on the performance of MLB
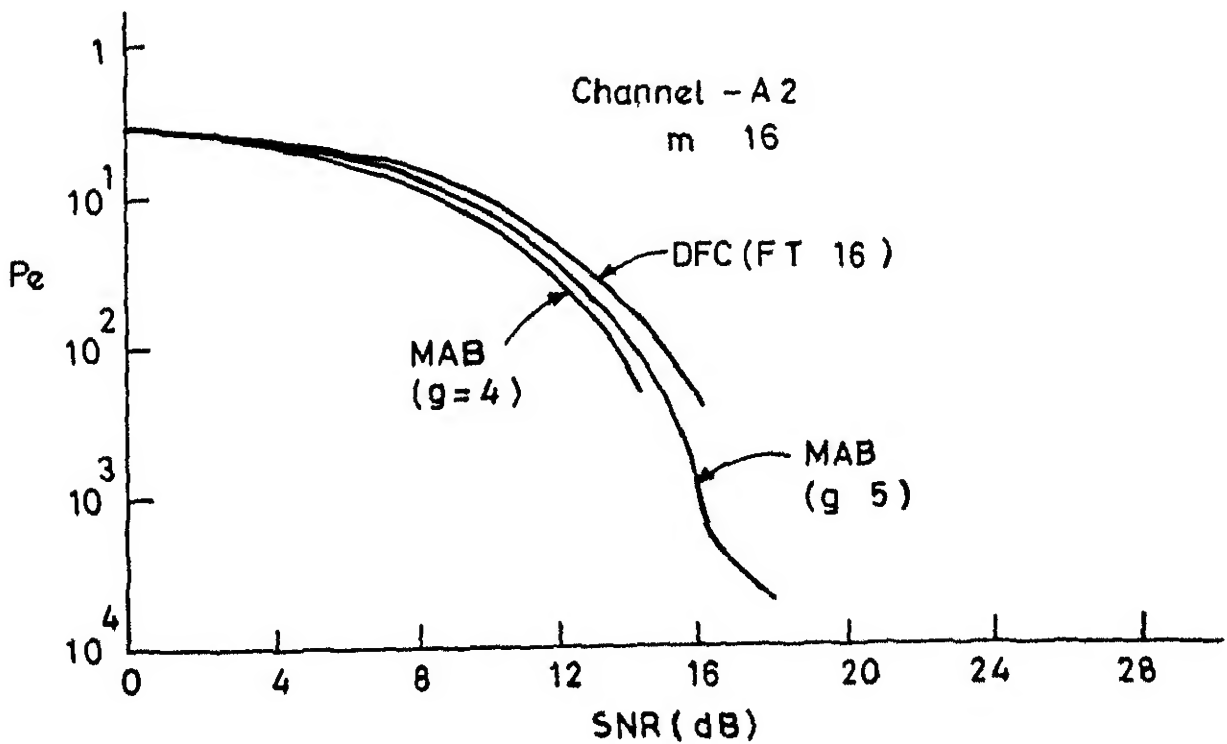
FIG 8 14 Effect of increased ISI on the performance of MAB

discrete time response values for various data ra es   The
performance curv s are shown in Figures 8 1   J 8 16   In
this case however BDT does not seem to enjoy noticeable
advantage over CSDI

Summarizing w evaluated th performance or various date
recovery schemes using simulation   It may be noted that c rc
must be exercised while making observations or while drawing
conclusions regarding a particular sch nc bas o on simulation
results   Thus with limited numb r of   examples and in the
absence of any theoretical justification it is difficult to
make any definite conclusion regarding the upericrity of
one schem over the other   However from our limited simula-
tion eyperiments on typical channels it app ars that by
choosing moderate block size and with incr ascd data ra e
over the chann 1 LB can be effectively implemented at a
reasonable computational and storage comple   co o red to
MLC which requires large amount of storage as ID >> 5g
Similarly we have obs rved that when the value of d_{min} is
close to 2 CCB or SCB works w ll compared to MLB with
relatively small amount of SNR loss   MAB also scens to be
effective compared to DFC although it requires more storage
Moreover it limits error propagation to a block length in
the worst situation — a feature absent in the case o DFC

CHAPTER   9

CONCLUSIONS AND SUGGESTIONS FOR FURTHER RESEARCH

In this thesis we have investigated the problem of
recovering data in block data transmission systems   We have
developed several data recovery schemes and compared their
performance with DR schemes available for continuous serial
data transmission systems   The data recovery problem has been
viewed as a finite deconvolution problem in the presence of
noise   Three deconvolution methods have been studied in
detail   They are   maximum likelihood   decision feedback   and
least squares deconvolution methods   The DR schemes developed
on the basis of these deconvolution methods may be viewed as
counterparts of the maximum likelihood   decision feedback   and
linear DR schemes in the context of CSDT

At this stage it may be appropriate to note the following
difference between BDT and CSDT systems   Due to practical
considerations   in CSDT systems   truncated number of observa-
tions are used for making decision regarding a transmitted data
symbol   although in theory we must use the entire observation
values   Thus the number of observations used for making deci-
sion determines the performance of a given DR scheme   In BDT
we control the data rate and data transmission format so as to
achieve better trade-off between error performance and computa-
tional complexity

Sali nt features o^ the study carricd out in this thesis and sugges ions for further research ar given in th following para raphs

The maximum likelihood recursiv d c nvolution al rithm dcvclopcd in Chaptei 3 is simpl to undcrstand compared to the convontiona Vitcrbi al jouithm approach which m_l^cs use of Bcllman s principle of optimality and statc r prcscntation The ML-RDA dcvcloped u ing Mit en s l m ia provi les nor insignt into the data rccovery problem compar o to th Vitcrbi algo- rithm application The computational compl xi ty and storage requircmcnts of ML RDA grows linearly with the block lengtn and cxponentially with he channel memory Simulation studie on some typical channels using iL RD for BDT and CoDT s/stems show tht ML-RDA for CoDT system requircs more storage and computetional complexity compared to ML-RDA for PDT systems in ordcr to achieve same error performance

W have further shown that in the case of birary dat_ it is possible to roducc both storage and computatior al requirc- ments of ML RDA by using rcsults from the theo~y of Boolcan and pseudo-Boolean functions Dcvclooment of similar methods for multil vel data using sonc of ne t hnicu s given in [72] for dealing with functions of discr te variabl s i an intcrcsting area for further rescarch

We have also shown that the method of minimizing pseudo Boolean functions provides a goo framewor! for developing ML-decoding sch mes for binary error correcting block codes Examples given in Ch pter 3 illustrate the usefuln ss of th method described We however have not stucied this aspect in detail The technique d velosed in this thesis can be extended to nonbinary codes al o A detailed study of codes over other fields using some of the r sults available in [72] is expected to prove usefu in a better understanding of ML decoding of linear as well as nonlinear block codes

The decision feedback schemes developed in Chapter 4 are computationally more attractive in s tuations where ML-RDA becomes uneconomical Unlike the conventional methods of deriving decision feedback data recovery schemes w have motivated the idea of decision f edback via Gaussian elimination method We have shown that th modified Austin approach can also be viewed as Gaussian eliminatior method with a nonlinear decision device

The DF-DR schemes for BDT systems restrict error propagation to atmost a block length whereas in the cas of DF-DP schemes for CSDT systers error propagation can extend to much longer durations However DR sch mes f r JDT systems require more storage compared to DR schemes for CSDT systems for achiev ng same probability of error performance

The DR scheme studied in Chapter 5 have the lea t computational and storage requirements compared to t cse studied in Chapters 3 and 4 However thi 1 only at t e exoense of probabiliy of error performarce However simu lation studies snow that ir the case of a gooa number or channels DR schemes Lased on k Circulant completions per orms well compared to DR schemes based on LS-deconvolution Je have shown that k Circulant completions of th channel convo lution matrix provides a more gen ral etting for studying deconvolution problems using di cr te Fourier transform A detailed study of optimum choice of k-Circulant completions according to some performance criteria needs further investi- gation

In Chapter 6 we have showed hat deconvolution using DFT and least squares deconvolution can be viewed as particular cases of square conpletions of the channel convolution atrix

Detailed anal/ is of he Steepest d scent algorithm in Chapter 7 shows that in the case of som examoles good estima- tes of the convergence parameter can be obtained using 1 - Circulant completions This analysis clearly explains the observation made by others that DFT domain fixcd tep steepest descent algorithm converges faster han the sa ple domain chemes Moreover w have shown that such estimat s can be used in the sample domain achiev goud conv roence

The simulation results given in Chapter 8 cov ¬ a nl
range of block lengths and typic l channel exa pl s   It is
interesting to note that the effec of incr asod ISI   among
the symbols of a block due to increased data rate ov r the
channel on the perforrance is not signiricant Simulation
studies also indicate that with appropriate ch ice of block
lengths BDT provides better tr de-off b twcen   rror   rformance
and processing c mplexity compar d to th av il blc DR schemes
for CSDT syste s   Howev r as imulation results arc always
based on a few examples n definite claims can be made regard-
ing the superiori y of BDT ov r CSDT

The maximum minimum distance and maximum dist nce ch nnels
obtain d in Chapter 8 show th t scme of the good channels are
partial response ch nnels How ver in the absence of any clear
definition of partial response channel [94] it is difficult to
show the connection between partial response na MMD or MD
channels This asp ct needs further investigation

Another asoect of the data r covery problem in which we
could not achieve much success is in obtaining go a bounds for
the error p rform nce of max mum likelihood ana d cision f d-
back sch mes The difficulty secms to be du to the nonlinear
nature of the DR schemes However w still f el that it must
b possible to obtain useful bounds for orobability of error
which in this case will be functions of block length and channel

memory   ihi   aspect requir s further investigation   Some
work in this direction has b en carried out by Klein and
Wolf [54]

In his thesis we have not considered the problem of
adapting the parameters of th data recovery schem s to the
channel variations   Depending on the nature of the channel
variation the design of th dat r covery schemes will change
The problem is interesting due to th presence of noise and our
limitation in recovering data without errors   Further investi-
gation is needed to understand various aspects of this problem

APPENDIX A

MITTEN S LEMMA

In this appendix we state and prove Mitten lemma [55]

Mitten s Lemma   Let $X = (x)$ be an arbitrary set and let $Y_1(x) = (y_1)$ $1 = 1\ 2$ n be   finite sequence o arbitrary sets whose definitions may depend on $x \varepsilon X$ Let $g_1(x\ y_1)$ $1 - 1\ 2$ n be a sequence of real valued functions for which $g_1^*(x) = \min[g_1(x\ y_1)|y_1 \varepsilon Y_1(x)]$ exists for $1 = 1\ 2$ n and all $x \varepsilon X$ Let $f(x)$ be an arbitrary function and let $g(x\ y_1$ $y_n) - h[f(x)\ g_1(x\ v_1)$ $g_n(x\ y_n)]$ be a real valued function for which $g^* = \min[g(x_1\ y_1$ $y_n)|$ $x \varepsilon X$ $y_1 \varepsilon Y_1(x)$ $1 = 1\ 2$ n]   ists and $h^*(x) = h[f(x)$ $g_1^*(x)$ $g_n^*(x)]$ exist for all $x \varepsilon X$ If $h (x) \leq g(x_1\ y_1$ $y_n)$ for ver/ $x \varepsilon X$ and $y_1 \varepsilon Y_1(x)$ $1 = 1\ 2$ n then $g = \min[h^*(x)|x \varepsilon X] = h$

Proof   The lemma can b proved by the method of contradiction For this purpose assum that there exists $x \varepsilon X$ and $y_1 \varepsilon Y_1(x)$ $1 = 1\ 2$ n such that $g(x\ y_1$ $y_n) < h$

But from the definition of minimum $h^* \leq h^*(x)$ and $h (x) \leq g(x\ y_1$ $y_n)$ by the assumed property of $h^*(x)$ Therefore

$$g(x\ y_1 \qquad y_n) < h^* \leq h^*(x) < g(x\ y_1 \qquad y_n)$$

which is a contradiction

APP I DIX B

k CIRCUL JT IATRICE

The purpose of this appen 1x 1 to list a rev properties of k-Circulant matrices [81] [82]

The k-Circulant matrices con idered in our study have the following structure

$$
C_k = \begin{bmatrix}
C_o & kC_{n-1} & kC_{n\ 2} & & kC_1 \\
C_1 & C_o & kC_{n\ 1} & & kC_2 \\
C_2 & C_1 & C_o & & \\
& & & & \\
& & & & \\
C_{1-1} & C_{n-2} & C_{n\ 3} & & C_o
\end{bmatrix}
$$

where $|k| = 1$ When $k = 1$ C_1 is a Circulant matrix $k = -1$ gives skew Circulant matrix Skew Circulants ar also known as negacyclic matrices [82]

It is convenient to study k-Circulant ith the help of the simple k-Circulan natrix

$$
\eta_k = \begin{bmatrix}
O & kI_1 \\
& \\
I_{n-1} & O_{n-1\ n-1}
\end{bmatrix}
$$

A matrix A is said to be r Circulant iff

$$A\eta_k = \eta_k A$$

in terms of η_k C_k can be expressed as

$$C_k = C_o I_n + C_1 \eta_k + C_2 \eta_k^2 + \quad\quad C_{n-1} \eta_k^{r-1} \quad\quad\quad (B\ 1)$$

S nce every k Circulant matrix commutes with η_k it is suffi-
cient to know the eigenvalu s and eig nvectors of η_k in order
to characterise and analys   Circulants

The eigenvalu s of η_k are the nth root of k for the
characteristic equation of η_k is given by

$$\lambda^n - k = 0$$

Let p be an nth root of k   Th n $p\omega^1$ $1 = 0\ 1$ $n-1$
where $\omega = e^{j2\pi/n}$ represents n roots of $\lambda^n - k = 0$

We now show that th v ctor

$$x = (1\ x^{-1}\ x^{-2}\quad\quad x^{-(n-1)})^T$$ wh re x is an nth
root of k is a eigenvector of η_k

A simple calculation shows that

$$\eta_k\ x = \begin{bmatrix} 1\ x^{-(n-1)} \\ 1 \\ x^{-1} \\ x^{-2} \\ x^{-(n-2)} \end{bmatrix} = x\ \underline{x}$$

Thus n roots of k generate n eigenvectors of η_1 Let F_k^{-1} bc a matrix with the n eigenv ctors of η_k as columns   It has the following form

$$
F_k^{-1} = \frac{1}{\sqrt{n}}
\begin{bmatrix}
1 & 1 & & 1 \\
p^{-1} & (p\omega)^{-1} & & (p\omega^{n-1})^{-1} \\
p^{-2} & (p\omega)^{-2} & & (p\omega^{n-1})^{-2} \\
& & & \\
& & & \\
p^{-(n-1)} & (p\omega)^{-(n-1)} & & (p\omega^{n-1})^{-(n-1)}
\end{bmatrix}
\quad (\text{B } 2)
$$

Note   that $\frac{1}{\sqrt{n}}$ is just a normalization factor used for making $F_k^{-1} \Gamma_k = I_n$

   Using B 2   it can b   shown that

$$
F_k \eta_k F_k^{-1} = \Lambda_k \qquad\qquad (B\ 3)
$$

where Λ_k is a diagonal matrix with $p\omega^i$ $i = 0\ 1$ $n-1$ as d agonal elements

Wh n $k = 1$ $F_k = F_1 = F$ the DFT matrix   is obtained   In virtue of this we shall refer   to F_k as κ-DFT m trix   In t rms of F we can write F_k^{-1} a

$$
F_k^{-1} = D^{-1} F^{-1} \qquad\qquad (B\ 4)
$$

where D is a diagonal matrix with p^i $i = 0\ 1$ $n-1$ as diagonal entries   Using (B 4) in (B 3) v obtain

$$F \, D \, \eta_k \, D^{-1} F^{-1} = \Lambda_k \qquad (B\ 5)$$

If C_1 is a Circulant matrix we know that

$$\Gamma \, C_1 \, \Gamma^{-1} = \Lambda_1 \qquad (B\ 6)$$

where Λ_1 is a diagonal matrix with eigenvalues of C_1 as diagonal entries

Comparing (B 6) and (B 5) we see that $D \, \eta_k \, D^{-1}$ represents a Circulant matrix

Post and pe multiplying B 1 by F_k^{-1} and F_k re pectively and using (B 3) we obtain

$$F_k \, C_k \, F_k^{-1} = C_0 I + C_1 \Lambda_k + C_2 \Lambda_k^2 + \quad C_{n-1} \Lambda_k^{n-1} \qquad (B\ 7)$$

From (B 7) we see that if $\lambda_i \quad i = 0,1 \quad n\,1$ are the eigenvalues of C_k

$$\lambda_i = \sum_{j=0}^{n-1} C_j (p\omega^i)^j \qquad (B\ 8)$$

(B 8) can also be rewritten as

$$\lambda_i = \sum_{j=0}^{n-1} (C_j p^j) \, \omega^{ij} \qquad (B\ 9)$$

(B 9) exhibits an interesting aspect eigenvalues of C_k can be determined by finding the eigenvalues of a Circulant matrix having firs column

$$(C_0 \ C_1 p \ C_2 p^2 \qquad C_{n-1} p^{n-1})^T \qquad (B\ 10)$$

This can also be seen directly u ing (B 7)   Usina (B 4) in (B 7) we get

$$FDC_kD^{-1}\Gamma^{-1} = \Lambda = diag(\Lambda_0 \lambda_1 \qquad \lambda_{n\ 1}) \qquad (B\ 11)$$

Therefore DC_kD^{-1} is a Circulant matrix   By little calculation we can see that the first column of this n trix is given by (B 10)

Finally consider inverse of a k-Circulant   ri ting (B 3) in terms of η_k^{-1} whenever η_k^{1} exists   Je get

$$\eta_k^{-1} = \Gamma_k^{1} \Lambda_k^{-1} \quad k$$

or $\qquad F_k\eta_k^{-1}F_k^{-1} = \Lambda_k^{-1}$

which shows that inverse of a k Circulant   whenever it exists is also a k-Circulant

REFERENCES

1   Nyquist H   Certain topics in telegraph transmission
    theory   Trans AIEE   vol 47 1928 pp 617-644

2   Clark A P   Adaptive detection of distorted digital
    signals   Radio Electron Eng   vol 40 1970 pp 107-119

3   Clark A P   A synchronous serial data transmission system
    using orthogonal groups of binary signal elements
    IEE Trans Commun   vol COM-19 1971 pp 1101-1110

4   Clark A P   Advanced data transmission s   ns   Pentech
    Press London 1977

5   Bennet J R and Davey J R   Data transmission   McGraw-
    Hill New York 1965

6   Lucky R W   Salz J and Weldon E J Jr   Principles of
    data communications   McGraw-Hill New York 1968

7   Proakis J G   Digital communications   McGraw Hill
    Tokyo 1983

8   Franks L E   Data communication Fundamentals of baseband
    transmission   Dowden Hutchinson and Ross Inc
    Pennsylvania 1974

9   Lucky R W   A survey of the communication theory litera-
    ture 1968-1973 IEE Trans Inform Theory vol IT-19
    1973 pp 725-739

10 Tufts D W   Nyquist s problem -The joint optimization of
    transmitter and receiver in pulse amplitude   modulation
    Proc IEEE vol 53 1965 pp 248-259

11 Tufts D W and Shnidman D A   Optimum waveforms subject
    to both energy and peak value constraints Proc IEEE
    vol 52 1964 pp 1002 1007

12 Smith J W   The joint optimization of transmitted signal
    and receiving filter for data transmission systems   Bell
    Syst Tech Journ   vol 44 1965 pp 2363-2392

13 Austin M E   Decision feedback equ lization for a gital communication over dispersive channels   I I T   Lin oln Lab   Lexington Mass   Tech Rept   437   1967

14 Lucky R W   Automatic equalization for dig_tal co unication   Bell Syst Tech Journ   vol 44 196␣ pp   47-588

15 Sha R T and Tang D T   A new cl ss o autor tic equali-zers   IBM Jl Res Dev   vol 1␣ 1972 p␣ ␣56-␣66

16 Chang R W and Ho E Y   On fast s art up data co munica tion systems using pseuc␣-random training sequences Bell ␣yst Tech Journ   vol ␣1 1972 ␣p ␣013 2␣2

17 Mueller K H and Spaulding D A   Cyclic e␣ualization - a new rapidly converging equalization technicu ␣or synchronous data communicatio␣ Bell ␣/ ␣ Tec␣ Journ vol 54   1975   pp 369-40␣

18 Koso␣ych O S and Pic␣holtz   Automatic equalization using successive overrelazation ␣terativ echni␣ue ␣ EE Trans Inform Theory vol IT 21 1975 pp 51 53

19 Chang R W   A new equalizer structure for fast start up digital communicstion   Bell ␣yst Tec␣ Journ   ␣ol 50 1971 pp 1969-2014

20 Kudin H R   Automatic equalization using transversal filters   IEEE Spectrum vol 4 1967 pp 53-59

21 Walzman T and Schwartz M   Automatic e ␣_liz tion using the discrete frequency domain   IE␣␣ Trar s Inf␣rm Theory vol IT-19 1973 pp 59 68

22 Walzman T and Schwartz M   A projected gradient method for automatic equalization in th frequency domain   IE E Tran Commun   vol CO ␣-21 1973 pp 144␣-1446

23 Lucky R W   Techniques for adaptive equalization of digital communication systcms   Bell ␣ys   Tech Journ vol 45 1966 pp 255-286

24 Gersho A   Adaptive equalization of highly disp rsive channels for data transmission   Bell Syst Tccb Journ vol 48 1969 pp 55-70

2␣ Proakis J G and Miller J H   An adaptive receiver for digital signalling through channels w␣th intersymbol interference IEL␣ Trans Inform Theory vol IT 15 1969 pp 484-497

26   Di Toro M J   Communication in time frequency spread media using adaptive equaliz tion   I E Proc   vol 5(1968 pp 1653-1679

27   Hirsch D and Wolf W J   A simple ada tive equali er for efficient data transmission   IEEE Tr n   Commun vol COM-18 1970 pp 5-12

28   Schonfeld T J and Schwartz M   Rapidly converging second order tracking algorithm for adaptive equaliza- tion   IEEE Trans Inform Theory vol IT-17 1971 pp 572-579

29   Salz J   Optimum mean-square decision feedback equali- zation Bell Syst Tech Journ   vol 52 1973 pp 1341-1373

30   Taylor D P   Nonlinear feedback equalizer employing a soft limiter   Electronics Letters vol 7 1971 pp 265-267

31   Taylor D P   The estimate feedback equalizer   a sub optimum nonlinear receiver   IE E Trans Commun   vol COM 21 1973 pp 979-990

32   Monsen P   Feedback equalization for fading dispersive channels   IEEE Tran Inform Theory vol IT-17 1971 pp 56-65

33   George D A   Bowen R P and Storey J P   An adaptive decision feedback equalizer   IEEE Trans Commun   vol COM-19 1971 pp 281-293

34   Duttweiler D L   Mazo J E and Messerschimtt D G   An upper bound on the error probability in decision feed- back equalization   IEEE Trans Inform Theory vol IT-20 1974 pp 490 497

35   alazar A C   Design of transmitter and receiver filters for decision feedb ck equalization   Bell Syst Tech Journ   vol 53 1974 pp 503-523

36   Lkanayake N and Taylor P   A decision feedback recei- ver structure for bandlimited nonlinear channels   IEEE Trans Commun   vol COM-29 1981 pp 539 548

37   Ling F and Proakis J G   Adaptive lattice decision feedback equalizers their performance and application to time invariant multipath channels   IEEE Trans Commun vol COM-33 1985 pp 348-356

38   Forney G D Jr   Maximum likelihood sequerce estimation
        of digital sequences in the resence of intersymbol int r-
        ferenco ICE   Trans Info m Theory vol IT-13 19/2 pp 363-378

39   Viterbi A J   Error bounds for convolutional codes and
        an asymptotically optimum decoding algorithm   ILEE Trans
        Inform Theory vol IT-13 1967 pp 201 269

40   Viterbi A J and Omura J K   Error bounus for intersym-
        bol interference channels   CISV Lectures No 229 The
        information theory approach to communications Edited by
        G Longo Springer Verlag 1978

41   Hayes J F   The Viterbi algorithm applied to digital
        transmission   IELE Communications society vol 13 1975
        pp 15-20

42   Forney G D Jr   The /iterbi algorithm   Proc   IEEE
        vol 61   1973 pp 268-_78

43   Kobaya hi H   A surv / of coding sch mes for transmi-
        ssion or recording of digital data   J "E Trans   Commun
        vol COM-19 1971 pp 1087-1100

44   Qureshi S U H and Newhall E E   An adaptive receiver for
        data transmission ov r time disper ive channels   IEEE
        Trans Inform Theory vol IT-19 1973 pp 448-457

45   Fredricsson S A   Joint optimization o transmitter and
        receiver filters in digital PAM systems with a Viterbi
        detector   ILE Trans Inform Theory vol IT-22 1976
        pp 200-209

46   Lee W U and Hill F S Jr   A raximum lilelihood sequence
        estimator with decision fe dback equalization   IEEE
        Trans Commun   vol COM-25 1977 pp 971-980

47   Falconer D D and Magee F R   Adaptiv channel memory
        truncation for maximum likelihood sequence estimation
        Bell Syst Tech Journ   vol 52 1973 pp 941-945

48   Fredricsson S A   Opt mum transmitting filter in digital
        PAM systems with a Viterbi d tector   IELE Trans Inform
        Theory, vol I1 20 1974 pp 479-489

49   Antonio C and K ong K   Further results on Viterbi
        equalizer   I E Trans Irrorm Theory vol IT-20 1974
        pp 764-767

50   Beare C T   The cho ce of the desir d impulse respons
in combined linear Viterbi al jorithm qualiz rs   JLEE
Trans Commun   vol COM-26 1978 pp 1301 1307

51   Kazakos D   Computational savings and implementation of
maximum likelihood d tectors   Ik k Trans Inf rm
Theory vol IT-24 1978 pp 124-126

52   Clark A P and Clayden M   Pseudo binary Vi erbi detec-
ctor   IFE Proc   vol 131 No 2 April 1984 pp 208-218

53   Clark A P and Ghani F   Detection processes for ortho-
gonal groups of digital signals ICFE Trans Commun
vol COM-21 1973 pp 907-915

54   Klein T J and Wolf J   On th use of hannel intro
duced redundancy for error corre tion   IE E Trans
Commun   vol COM-19 1971 pp 396-402

55   Mitten L G   Composition principles for synthesis of
optimal multistage proce ses   Operations Reseaich
vol 12 1964 pp 610-619

56   Nemhauser G L   Dyn mic programming   John Wiley and
Sons Inc New York 1966

57   Hammer P L and Rudeanu S   Boolean m thods in opera-
tions research   Spr nger Verlag New York 1968

58   Wang C L   B st least square app oximation b/ minimum
property of Fourier expansions   Theory f proxima-
tions with applications Ed Alan G L and Sahancy B N
Academic Press 1976 pp 86-110

59   Hunt B R   The inverse problem of radiography   Math
Biosciences vol 8 1970 pp 161-179

60   Hunt B R   Deconvolution of linear systems by con-
strained regre sion and i s relationship to Wiener
theory   IEEE Trans Auto Contr   vol AC-17 1972
pp 703-705

61   Commenges D   The deconvolution problem fast algorithms
including the preconditioned conjugate-gradient to com-
pute a MAP estimator ICuC Trans Auto Contr vol AC-
29 1984 pp 229-243

62   Rosenfield A and Kac C   Digital picture processing
Second dition vol 1 Academic Press New York 1982

63   Dines K A and Kac C   Constrained least squres
      filtering IE_E Trans Accoust Speech and   al
      Process   vol ASSP 25 1977 po 346-350

64   Picchi G and Prati G   Self orthogonalizing
      adaptive equalization in the discret frequency domain
      IFEL Trans Commun   vol COM-32 No 4   April 1984
      pp 371-379

65   Pratt W K   Fast computational techniques for pseudo-
      inverse and Wiener image restoration   IEEE Trans
      Comouters vol C 26 1977 pp 571-580

66   Narayan S S and Peterson A M   Frequency domain least
      mean square algorithm   ILEE Proc   ol 69 19u1
      pp 124-126

67   Arya V < and Aggarwal J K   Deconvolution of seismic
      data   Hutchinson Ross Publishing Company Stroudsburg
      1982

68   Jansson P A   Deconvolution with applications in
      spectroscopy   Academic Press Inc New York 1984

69   Phillips D L   A technique for the numerical solution
      of certain integral equations of first kina   J Ass
      Comput Mach   vol 9 1962 pp 84-97

70   Twomey S   The application of numerical filtering to the
      solution of integral equations of the first kind encoun-
      tered in direct sensing measurements   J Fr nk Inst
      vol 279 1965 pp 95 109

71   Rice R B   Inverse convolution filters   Geophysics
      vol 27 1961 pp 4-18

72   Davio M Deschamps J P and Andre T   Discrete ana
      swi ching functions   McGraw Hill New York 1978

73   Wolf J K   Efficient maximum likelihood decoding of
      lin ar block codes using a trellis   IEEE Trans Inform
      Theory vol IT-24 1978 pp 76-80

74   Ralston A and Rabinowitz P   A first course in numerical
      analysis , McGraw-Hill Tokyo 1978

75   Trench W F   An algorithm for the inversion of finite
      Toeplitz matrices   J Soc Indust Aopli Math   vol 12
      1964 pp 515-522

76   Durbin J   Efficient estimation of param ters in moving-average models Biometrika vol 46 1959 pp 306-316

77   Bennet W R   Synthesis of acti/e n tworks Proc of the s/moosium on modern n twork synthesis Polytechnique Institute of Brookl/n 1955 pp 45-61

78   Trench V F   Inv rsion of Toeolitz band matrices lath Computation vol 28 1974 pp 1069-1095

79   Dickinson R J   cff cient solution of linear equations with banded Toeplitz matrice   EcE Trar s Accoust Speech Signal Process vol A55P-27 1979 pp 421-423

80   Jain A K   F st inversion of banded To~oli z matrices by circular decompositions IEEE T ans Accoust Speech Signal Process vol A SP 26 1978 pp 121-126

81   Clin R E Plemmons P J anc Jor G Gen ralized inverse of certain Toeolitz matrices LA and its applications vol 8 1974 pp 25-33

82   Davis P J Circulant matrices John Wiley and Sons New York 1979

83   Cool y J V Lewis P and Welch P Applicatior of the fast Fourier transform to the computation of Fourier integrals Fourier series and convolutional integrals IE L Trans Audio Fl ctroacoustics vol AU-15 1967 pp 79-84

84   Cooley J W and Tukey J W An algori l or the machine calculation of complex Fourier eries lath Computation vol 19 1965 pp 297-301

85   Penrose R   A generalized inv rse for matrices Proc Cambridge Phil Soc vol 51 1955 pp 406 413

86   Penrose R   On best approximate solutions of linear matrix equations Proc Cambridg Phil Soc vol 52 1956 pp 17-19

87   Campbell S L and Meyer C D Jr Generalized inv rses of linear transformations Pitman Publishers London 1979

88   Godard D   Channel equalization using a Kalman filter for fast data transmission IBM Jl Res D v vol 18 1974 pp 267-273

89   McCormick G P   Nonlinear progr mming   heo y
      algorithms and applications   John liley and Sons
      New York   1983

90   Goldstein A A   Constructive real analysis   Harp r nd
      Row New York   1967

91   Rosen J B   The gradient projection metho l for nonlinear
      programming Part I   SIAI J Appl Nath   1960   pp 181-218

92   Kobayashi H   Application of Hestens-S iefel algorithm
      to channel equalization   Con rence record IELC Inter
      national Conference on Commurications   lon real   1971

93   Kuhn H / and Tucker A /   Linear irequali ies and
      related s/stems   Annals of Mathematical Studies   No 38
      Princeton Univ Pre s 1056

94   Kretzmer E R   (en ali ation o a techi que for binary
      data communic tion   I___ Tr ns Commun   vol CO\ 14 1966
      pp 67-68

95   Burkhardt H and Barbosa L C   Cortributions to the
      application of the Viterbi algorithm   ILEt Trans Inform
      Theory vol IT-31   1985   pp 626-634

95066

E-1906-D-POO FIN